Homework 3: Solutions

October 24, 2015

1 Sequences of Poissons

(a) From the definition of convergence in probability and the fact that $\mathbb{P}[|X_n| > \varepsilon] \leq \mathbb{P}[X_n \neq 0]$ for all $\varepsilon > 0$, it is enough to show that $\mathbb{P}[X_n = 0] \to 1$. The pmf of a Poisson$(1/n)$ random variable is

$$\mathbb{P}[X_n = k] = \frac{(1/n)^k}{k!} e^{-1/n}, \quad k = 0, 1, 2, \ldots.$$

Therefore, $\mathbb{P}[X_n = 0] = e^{-1/n} \to 1$, which proves that $X_n \xrightarrow{P} 0$.

(b) Since Y only takes nonnegative integer values, it is still enough to show that $\mathbb{P}[Y_n = 0] \to 1$. We have

$$\mathbb{P}[Y_n = m] = \mathbb{P}[X_n = m/n] = \frac{(1/n)^{m/n}}{(m/n)!} e^{-1/n}, \quad \text{where } m = kn, \quad k = 0, 1, 2, \ldots.$$

Therefore, $\mathbb{P}[Y_n = 0] = e^{-1/n} \to 1$, which proves that $Y_n \xrightarrow{P} 0$.

2 Rademacher with a twist

Let Z_n be a Bernoulli$(1/n)$ random variable independent of X. Then X_n can be written as

$$X_n = \begin{cases} X, & \text{if } Z_n = 0 \\ e^n, & \text{if } Z_n = 1 \end{cases}.$$

(a) For all $\varepsilon > 0$,

$$\begin{aligned} \mathbb{P}[|X_n - X| > \varepsilon] &= (1 - 1/n)\mathbb{P}[|X_n - X| > \varepsilon | Z_n = 0] + \frac{1}{n}\mathbb{P}[|X_n - X| > \varepsilon | Z_n = 1] \\ &= (1 - 1/n)\mathbb{P}[0 > \varepsilon | Z_n = 0] + \frac{1}{n}\mathbb{P}[|e^n - X| > \varepsilon | Z_n = 1] \leq \frac{1}{n}, \end{aligned}$$

which implies that $X_n \xrightarrow{P} X$.

(b) It follows from part (a) that $X_n \xrightarrow{d} X$.

(c) X_n does not converge to X in mean square, because

$$\begin{aligned} \mathbb{E}[(X_n - X)^2] &= (1 - 1/n)\mathbb{E}[(X_n - X)^2 | Z_n = 0] + \frac{1}{n}\mathbb{E}[(X_n - X)^2 | Z_n = 1] \\ &= \frac{1}{n}\mathbb{E}[(e^n - X)^2] = \frac{1}{n}\Big(\frac{1}{2}(e^n - 1)^2 + \frac{1}{2}(e^n + 1)^2\Big) \to \infty. \end{aligned}$$

3 Bugs in code

The central limit theorem states that

$$\frac{Y - n\mathbb{E}[X]}{\sqrt{n\text{Var}[X]}} \xrightarrow{d} N(0,1).$$

So it is reasonable to approximate $\mathbb{P}[Y < 90]$ as

$$\mathbb{P}[Y < 90] = \mathbb{P}\left[\frac{Y - n\mathbb{E}[X]}{\sqrt{n\text{Var}[X]}} < \frac{90 - n\mathbb{E}[X]}{\sqrt{n\text{Var}[X]}}\right] \approx \Phi\left(\frac{90 - n\mathbb{E}[X]}{\sqrt{n\text{Var}[X]}}\right) = \Phi(-1) \approx 0.16.$$

4 Maxima of i.i.d. Gaussians

(a) From the union bound and the Chernoff bound, for all $\varepsilon > 0$,

$$\begin{aligned}\mathbb{P}\Big[\max_{1\le i\le n} \ge (1+\varepsilon)\sqrt{2\log n}\Big] &= \mathbb{P}\Big[\cup_{i=1}^n \left\{X_i \ge (1+\varepsilon)\sqrt{2\log n}\right\}\Big] \\ &\le n\mathbb{P}\Big[X_1 \ge (1+\varepsilon)\sqrt{2\log n}\Big] \\ &\le n\exp\Big(-\frac{(1+\varepsilon)^2 2\log n}{2}\Big) = \frac{1}{n^{2\varepsilon+\varepsilon^2}} \to 0.\end{aligned}$$

(b) Let $v \triangleq t - x$, then $t = v + x$. Using the fact that $(x+v)^2 \le (1+\delta^{-1})v^2 + (1+\delta)x^2$ for all $\delta > 0$ (from the hint, which can be proved by AM-GM inequality), and the fact that $\int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t}{2\sigma^2}}\,\mathrm{d}t = \frac{1}{2}$,

$$\begin{aligned}\mathbb{P}[X \ge x] &= \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{(v+x)^2}{2}}\,\mathrm{d}t \\ &\ge e^{-\frac{(1+\delta)x^2}{2}} \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{(1+\delta^{-1})(t-x)^2}{2}}\,\mathrm{d}t \\ &= e^{-\frac{(1+\delta)x^2}{2}} \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{(1+\delta^{-1})u^2}{2}}\,\mathrm{d}u = \frac{1}{2\sqrt{1+\delta^{-1}}} e^{-\frac{(1+\delta)x^2}{2}}.\end{aligned}$$

(c) From the independence among X_i's, for $0 < \varepsilon < 1$,

$$\begin{aligned}\mathbb{P}\Big[\max_{1\le i\le n} X_i \le (1-\varepsilon)\sqrt{2\log n}\Big] &= \Big(1 - \mathbb{P}\Big[X_1 > (1-\varepsilon)\sqrt{2\log n}\Big]\Big)^n \\ \text{(using the lower bound in part (b))} \quad &\le \left(1 - \frac{1}{2\sqrt{1+\delta^{-1}}} e^{-(1+\delta)(1-\varepsilon)^2\log n}\right)^n \\ &= \left(1 - \frac{1}{2\sqrt{1+\delta^{-1}}} \frac{1}{n^{(1+\delta)(1-\varepsilon)^2}}\right)^n \\ \text{(setting } \delta = \frac{\varepsilon}{1-\varepsilon} > 0\text{)} \quad &= \left(1 - \frac{1}{2\sqrt{1/\varepsilon}} \frac{1}{n^{1-\varepsilon}}\right)^n \\ \text{(using the fact that } 1 - x \le e^{-x}\text{)} \quad &\le \exp\left(-\frac{1}{2\sqrt{1/\varepsilon}} n^\varepsilon\right) \to 0.\end{aligned}$$

For $\varepsilon \geq 1$,

$$\mathbb{P}\Big[\max_{1\leq i\leq n} X_i \leq (1-\varepsilon)\sqrt{2\log n}\Big] \leq \mathbb{P}\Big[\max_{1\leq i\leq n} X_i \leq 0\Big] = (\mathbb{P}[X_1 \leq 0])^n = 1/2^n \to 0.$$

Combining the results in part (a) and part (c), we conclude that

$$\frac{\max_{1\leq i\leq n} X_i}{\sqrt{2\log n}} \xrightarrow{P} 1.$$

5 Gaussians and MMSE estimation

(a) Since $V + W = 5X + Y - 2Z + 5$, we have

$$\mathbb{E}[V+W] = 5+1-2+5 = 9, \quad \mathrm{Var}[V+W] = 25+1+4 = 30.$$

$V + W$ is a Gaussian random variable, because X, Y, Z are jointly Gaussian.

(b) Let $U = \begin{bmatrix} V \\ W \end{bmatrix}$. Then $\mathbb{E}[U] = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$, and $\mathrm{Cov}[U] = \begin{bmatrix} 5 & 6 \\ 6 & 13 \end{bmatrix}$. The characteristic function of U is

$$\varphi_U(t) = \exp\left(it^T\mathbb{E}[U] - \frac{1}{2}t^T\mathrm{Cov}[U]t\right).$$

(c)

$$\widehat{\mathbb{E}}[V|W] = \mathbb{E}[V] + \frac{\mathrm{Cov}(V,W)}{\mathrm{Var}[W]}(W - \mathbb{E}[W]) = 3 + \frac{6}{13}(W-6).$$

(d) $\widehat{\mathbb{E}}[V|W]$ is also the optimal nonlinear estimator, because V and W are jointly Gaussian.

(e) Let

$$A = a_1(X - \mathbb{E}X) + a_2(Y - \mathbb{E}Y)$$
$$B = b_1(X - \mathbb{E}X) + b_2(Y - \mathbb{E}Y).$$

Then we need to determine a_1, a_2, b_1, b_2 from the requirement on the covariance matrix:

$$\mathrm{Var}[A] = a_1^2 + a_2^2 = 3$$
$$\mathrm{Var}[B] = b_1^2 + b_2^2 = 5$$
$$\mathrm{Cov}(A,B) = a_1b_1 + a_2b_2 = 2.$$

Solving the above system of equations, we get

$$a_1 = a_2 = \sqrt{\frac{3}{2}}, \quad b_1 = \frac{2\sqrt{6}-\sqrt{66}}{6}, \quad b_2 = \frac{2\sqrt{6}+\sqrt{66}}{6}.$$