ECE 534 RANDOM PROCESSES **FALL 2015**
PROBLEM SET 3 **Due beginning of class, Thu, Oct 15**

3. Limits and MMSE Estimation

Assigned reading: Chapters 2 and 3 of the ECE 534 course notes. Be sure to at least skim through the problems at the end of the chapters.

Problems to be handed in:

1 Sequences of Poissons

For each $n = 1, 2, \ldots$, let X_n be a Poisson$(1/n)$ random variable.

(a) Prove that $X_n \xrightarrow{P} 0$.

(b) Let $Y_n = nX_n$. Prove that $Y_n \xrightarrow{P} 0$.

2 Rademacher with a twist

Let X be a Rademacher random variable, i.e., $\mathbb{P}[X = +1] = \mathbb{P}[X = -1] = 1/2$. Define a sequence of random variables $\{X_n\}_{n=1}^{\infty}$ as follows:

$$X_n = \begin{cases} X, & \text{with probability } 1 - \frac{1}{n} \\ e^n, & \text{with probability } \frac{1}{n}. \end{cases}$$

Prove or disprove the following statements:

(a) $X_n \xrightarrow{P} X$.

(b) $X_n \xrightarrow{d} X$.

(c) $\lim_{n\to\infty} \mathbb{E}[(X_n - X)^2] = 0$.

3 Bugs in code

We have a software project consisting of $n = 100$ pages of code. Let X_i be the number of bugs on the ith page of code. Suppose that the X_i's are i.i.d. Poisson(1) random variables. Let $Y = \sum_{i=1}^{n} X_n$ be the total number of bugs. Use the Central Limit Theorem to approximate $\mathbb{P}[Y < 90]$.

4 Maxima of i.i.d. Gaussians

The goal of this problem is to prove the following statement: If $X_1, X_2, \ldots$ are i.i.d. $N(0, 1)$ random variables, then

$$\frac{\max_{1\le i\le n} X_i}{\sqrt{2\log n}} - 1 \xrightarrow{P} 0. \tag{1}$$

(a) Prove that

$$\mathbb{P}\left[\max_{1\le i\le n} X_i \ge (1+\varepsilon)\sqrt{2\log n}\right] \xrightarrow{n\to\infty} 0, \qquad \forall \varepsilon > 0.$$

Hint: Estimate $\mathbb{P}[X \ge t]$ for $X \sim N(0, 1)$ and for all $t > 0$ via the Chernoff bound, and combine this estimate creatively with the union bound.

(b) Prove the following lower bound on the Gaussian tail: Let $X \sim N(0,1)$. Prove that, for any $x \in \mathbb{R}$ and any $\delta \in (0,1]$,

$$\mathbb{P}[X \geq x] \geq \frac{1}{2\sqrt{1+\delta^{-1}}} e^{-(1+\delta)x^2/2}.$$

Hint: Write the probability as an integral and use $(x+v)^2 \leq (1+\delta^{-1})v^2 + (1+\delta)x^2$.

(c) Using the result of part (b) with a clever choice of δ, prove that

$$\mathbb{P}\left[\max_{1 \leq i \leq n} X_i \leq (1-\varepsilon)\sqrt{2\log n}\right] \xrightarrow{n\to\infty} 0, \qquad \forall \varepsilon > 0$$

and thus complete the proof of (1).

5 Gaussians and MMSE estimation

Let X, Y, Z be i.i.d. $N(1,1)$ random variables. Consider the random variables

$$V = 2X + Y \qquad \text{and} \qquad W = 3X - 2Z + 5.$$

(a) Find the mean and the variance of $V + W$. Is $V + W$ a Gaussian random variable? Why or why not?

(b) Find the characteristic function of the random vector $\begin{pmatrix} V \\ W \end{pmatrix}$.

(c) Find the linear MMSE estimator $\widehat{\mathbb{E}}[V|W]$ of V given W.

(d) Is this an optimal nonlinear estimator? Why or why not?

(e) The zero-mean random variables $X - \mathbb{E}X$, $Y - \mathbb{E}Y$, $Z - \mathbb{E}Z$ are the inputs to a black box. There are two outputs, A and B, and we wish to design the black box so that the covariance matrix of the output vector $\begin{pmatrix} A \\ B \end{pmatrix}$ has the form

$$\mathrm{Cov}(A,B) = \begin{pmatrix} 3 & 2 \\ 2 & 5 \end{pmatrix}.$$

Find expressions for A and B in terms of the zero-mean inputs to the black box to guarantee this. (The answer is not necessarily unique.)