Homework 2: Solutions

October 8, 2015

1 The joys of symmetry

(a) We prove the following proposition. The claim that Z is symmetric then follows by induction.

Proposition 1. *If Y is a symmetric random variable and X is a Rademacher random variable independent of Y, then for any real constant r, the random variable $Y + rX$ is symmetric.*

Proof. Let $Z \triangleq Y + rX$. For any $z \in \mathbb{R}$,

$$\begin{aligned}\mathbb{P}[Z \le z] &= \frac{1}{2}\mathbb{P}[Y + r \le z] + \frac{1}{2}\mathbb{P}[Y - r \le z] \\ &= \frac{1}{2}\mathbb{P}[-Y \le z - r] + \frac{1}{2}\mathbb{P}[-Y \le z + r] = \mathbb{P}[-Z \le z],\end{aligned}$$

where the second step is due to the symmetry of Y. This proves that $Y + rX$ is symmetric. □

Now we prove that $\mathbb{E}[Z^4] \le 3(\mathbb{E}[Z^2])^2$. We have

$$\mathbb{E}[Z^2] = \mathbb{E}\Big[\sum_{i=1}^n r_i^2\Big] + \mathbb{E}\Big[2\sum_{i=1}^n \sum_{j>i} r_i r_j X_i X_j\Big] = \sum_{i=1}^n r_i^2.$$

Furthermore,

$$\begin{aligned}\mathbb{E}[Z^4] &= \mathbb{E}\Big[\Big(\sum_{i=1}^n r_i^2\Big)^2\Big] + \mathbb{E}\Big[4\Big(\sum_{i=1}^n r_i^2\Big)\Big(\sum_{i=1}^n \sum_{j>i} r_i r_j X_i X_j\Big)\Big] + \mathbb{E}\Big[4\Big(\sum_{i=1}^n \sum_{j>i} r_i r_j X_i X_j\Big)^2\Big] \\ &= \Big(\sum_{i=1}^n r_i^2\Big)^2 + 0 + 4\sum_{i=1}^n \sum_{j>i} r_i^2 r_j^2 \le \Big(\sum_{i=1}^n r_i^2\Big)^2 + 2\Big(\sum_{i=1}^n r_i^2\Big)\Big(\sum_{j=1}^n r_j^2\Big) = 3\mathbb{E}[Z^2].\end{aligned}$$

(b) For any $t \ge 0$,

$$\begin{aligned}\mathbb{P}[S|X| \le t] &= \frac{1}{2}\mathbb{P}[|X| \le t] + \frac{1}{2}\mathbb{P}[-|X| \le t] = \frac{1}{2}\mathbb{P}[|X| \le t] + \frac{1}{2} = \frac{1}{2}(\mathbb{P}[X \le t] - \mathbb{P}[X < -t]) + \frac{1}{2} \\ &= \frac{1}{2}\mathbb{P}[X \le t] + \frac{1}{2}(\mathbb{P}[X \ge -t] = \mathbb{P}[X \le t].\end{aligned}$$

For any $t < 0$,

$$\mathbb{P}[S|X| \le t] = \frac{1}{2}\mathbb{P}[|X| \le t] + \frac{1}{2}\mathbb{P}[-|X| \le t] = \frac{1}{2}\mathbb{P}[|X| \ge -t] = \frac{1}{2}(\mathbb{P}[X \ge -t] + \mathbb{P}[X \le t]) = \mathbb{P}[X \le t].$$

Thus we have shown that for all $t \in \mathbb{R}$, $\mathbb{P}[S|X| \le t] = \mathbb{P}[X \le t]$.

2 Indicator functions for fun and profit

(a) Since X is nonnegative, for any $t > 0$,

$$\mathbb{E}[X \wedge t] = \mathbb{E}[(X \wedge t)\mathbf{1}\{X > t\}] + \mathbb{E}[(X \wedge t)\mathbf{1}\{X \le t\}] \ge t\mathbb{P}[X > t] + 0.$$

It follows that $\mathbb{P}[X > t] \le \mathbb{E}[X \wedge t]/t$ for any $t > 0$.

(b) We have

$$\mathbb{P}[X > t] = \mathbb{P}[e^X > e^t] \le \mathbb{E}[e^X]/e^t,$$

where the last step follows from the Markov inequality.

(c) Since $X \in [0,1]$, for any $t > 0$,

$$\mathbb{E}X = \mathbb{E}[X\mathbf{1}\{X > t\}] + \mathbb{E}[X\mathbf{1}\{X \le t\}] \le \mathbb{E}[1 \cdot \mathbf{1}\{X > t\}] + \mathbb{E}[t \cdot \mathbf{1}\{X \le t\}] \le \mathbb{P}[X > t] + t.$$

(d) Let $B = \cup_i A_i$. From the Cauchy-Schwarz inequality,

$$\left(\mathbb{E}\Big[\mathbf{1}_B \sum_{i=1}^n \mathbf{1}_{A_i}\Big]\right)^2 \le \mathbb{E}\big[\mathbf{1}_B^2\big]\mathbb{E}\left[\Big(\sum_{i=1}^n \mathbf{1}_{A_i}\Big)^2\right]. \tag{1}$$

From the hint,

$$\text{LHS of (1)} = \left(\mathbb{E}\Big[\sum_{i=1}^n \mathbf{1}_{A_i}\Big]\right)^2 = \sum_{i=1}^n \sum_{j=1}^n \mathbb{P}[A_i]\mathbb{P}[A_j].$$

Moreover,

$$\text{RHS of (1)} = \mathbb{P}[B]\mathbb{E}\Big[\sum_{i=1}^n \sum_{j=1}^n \mathbf{1}_{A_i}\mathbf{1}_{A_j}\Big] = \mathbb{P}[\cup_i A_i] \sum_{i=1}^n \sum_{j=1}^n \mathbb{P}[A_i \cap A_j].$$

The claim follows from the fact that $\sum_{i=1}^n \sum_{j=1}^n \mathbb{P}[A_i \cap A_j] \ge \max_{1 \le i \le n} \mathbb{P}[A_i] > 0$.

3 Lower bounds for lower tails

(a) For any $r \in [0,1]$,

$$\mathbb{E}[X] = \mathbb{E}[X\mathbf{1}\{X \ge r\mathbb{E}X\}] + \mathbb{E}[X\mathbf{1}\{X < r\mathbb{E}X\}] \le \sqrt{\mathbb{E}[X^2]\mathbb{P}[X \ge r\mathbb{E}X]} + r\mathbb{E}[X],$$

where the second step is due to the Cauchy-Schwarz inequality. Since $0 < \mathbb{E}X < \infty$, we have

$$\mathbb{P}[X \ge \mathbb{E}X] \ge (1-r)^2(\mathbb{E}X)^2/\mathbb{E}[X^2].$$

(b) Let $M_i \triangleq |X_i|$, $i = 1, \ldots, n$, and let $S_1, \ldots, S_n$ be independent Rademacher random variables that are independent of X_i's. Since $X_1, \ldots, X_n$ are independent symmetric random variables, from Problem 1(b), we know that $(X_1, \ldots, X_n)$ and $(S_1M_1, \ldots, S_nM_n)$ have the same joint distribution. We thus only need to show the inequality with X_i replaced with S_iM_i, $i = 1, \ldots, n$.

From the law of total probability, it suffices to show that

$$\mathbb{P}\Big[\Big(\sum_{i=1}^n S_iM_i\Big)^2 \ge r\sum_{i=1}^n (S_iM_i)^2 \Big| M_1 = m_1, \ldots, M_n = m_n\Big] \ge \frac{(1-r)^2}{3}, \quad \forall m_1, \ldots, m_n. \tag{2}$$

Since S_i's and M_i's are independent,

$$\text{LHS of (2)} = \mathbb{P}\Big[\Big(\sum_{i=1}^{n} S_i m_i\Big)^2 \geq r\sum_{i=1}^{n} m_i^2\Big] = \mathbb{P}\Big[\Big(\sum_{i=1}^{n} S_i m_i\Big)^2 \geq r\mathbb{E}\Big[\Big(\sum_{i=1}^{n} S_i m_i\Big)^2\Big]\Big]$$
$$= \mathbb{P}\big[Z^2 \geq r\mathbb{E}[Z^2]\big] \geq (1-r)^2\frac{(\mathbb{E}Z^2)^2}{\mathbb{E}[Z^4]} \geq \frac{(1-r)^2}{3},$$

where we have defined $Z \triangleq \sum_{i=1}^{n} S_i m_i$, and used the results of Problem 3(a) and Problem 1(a).

4 The importance of lowered expectations

(a) First, we show that $X_n \xrightarrow{p.} X \implies \lim_{n\to\infty} \mathbb{E}[|X_n - X| \wedge 1] = 0$.

If $X_n \xrightarrow{p.} X$, then $X_n - X \xrightarrow{d.} 0$. Since $|X_n - X| \wedge 1$ is a continuous and bounded function of $X_n - X$, from the property of convergence in distribution, we have $\lim_{n\to\infty} \mathbb{E}[|X_n - X| \wedge 1] = 0$.

(b) Second, we show that $\lim_{n\to\infty} \mathbb{E}[|X_n - X| \wedge 1] = 0 \implies X_n \xrightarrow{p.} X$.

For any $\varepsilon > 0$, we have

$$\mathbb{E}[|X_n - X| \wedge 1] \geq \mathbb{E}[(|X_n - X| \wedge 1)\mathbf{1}\{|X_n - X| \geq \varepsilon\}] \geq (\varepsilon \wedge 1)\mathbb{P}[|X_n - X| \geq \varepsilon] \geq 0.$$

Therefore, if $\lim_{n\to\infty} \mathbb{E}[|X_n - X| \wedge 1] = 0$, then $\lim_{n\to\infty} \mathbb{P}[|X_n - X| \geq \varepsilon] = 0$ for all $\varepsilon > 0$, which implies that $X_n \xrightarrow{p.} X$.

5 Probably but not surely

The following partial converse of the Borel-Cantelli Lemma is essential to prove this result.

Lemma 1. *For independent events $A_1, A_2, \ldots$, if $\sum_{n=1}^{\infty} \mathbb{P}[A_n] = \infty$, then $\mathbb{P}[A_n \ i.o.] = 1$.*

From the definition of almost sure convergence, to prove the existence of $c_1, c_2, \ldots$ such that the sequence $c_n X_n$ does not converge to 0 almost surely, it suffices to show that

Proposition 2. *For some $\varepsilon_0 > 0$, there exist $c_1 \geq c_2 \geq \ldots > 0$, where $\lim_{n\to\infty} c_n = 0$, such that*

$$\mathbb{P}[|c_n X_n| \geq \varepsilon_0 \ i.o.] = 1.$$

To prove that for the above choice of $c_1, c_2, \ldots$, the sequence $c_n X_n$ converges to 0 in probability, it suffices to show that

Proposition 3. *If $\lim_{n\to\infty} c_n = 0$, then for all $\varepsilon > 0$,*

$$\lim_{n\to\infty} \mathbb{P}[|c_n X_n| > \varepsilon] = 0.$$

We start with Proposition 3, which is easier to prove.

Proof of Proposition 3. For any $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \mathbb{P}[|c_n X_n| > \varepsilon] = \lim_{n\to\infty} \mathbb{P}[|X_n| > \varepsilon/|c_n|] = 0,$$

where the first equality is due to the fact that $\mathbb{P}[|c_n X_n| > \varepsilon] = \mathbb{P}[|X_n| > \varepsilon/|c_n|]$, and the second equality follows from the fact that $\varepsilon/|c_n| \to \infty$ as $c_n \to 0$ and the property of the cumulative distribution function. □

Now we prove Proposition 2.

Proof of Proposition 2. Let X be an i.i.d. copy of $X_1, X_2, \ldots$. Fix some $\varepsilon_0 > 0$, and let

$$c_n \triangleq \inf\{c > 0 : \mathbb{P}[c|X| \geq \varepsilon_0] \geq 1/n\}.$$

Let $g(c) \triangleq \mathbb{P}[c|X| \geq \varepsilon_0]$. It can be checked that $c \mapsto g(c)$ is non-decreasing and right-continuous. Therefore, $c_n \in \{c > 0 : \mathbb{P}[c|X| \geq \varepsilon_0] \geq 1/n\}$, namely,

$$c_n = \min_c \{c > 0 : \mathbb{P}[c|X| \geq \varepsilon_0] \geq 1/n\}. \tag{3}$$

It follows that

$$c_1 \geq c_2 \geq \ldots > 0. \tag{4}$$

We then prove $\lim_{n\to\infty} c_n = 0$ by contradiction. Suppose that for some $\delta > 0$, $c_n > \delta$ for all n. Then by (3), $\mathbb{P}[\delta|X| \geq \varepsilon_0] < 1/n$ for all n. This implies that $\mathbb{P}[|X| \geq \varepsilon_0/\delta] = 0$, which contradicts the assumption that $\mathbb{P}[|X| \geq t] \geq \mathbb{P}[|X| > t] > 0$ for all $t > 0$. Together with (4), this proves that $\lim_{n\to\infty} c_n = 0$.

Finally, we use Lemma 1 to complete the proof. Let $A_n \triangleq \{c_n|X_n| \geq \varepsilon_0\}$. Then by (3),

$$\sum_n^\infty \mathbb{P}[A_n] \geq \sum_n^\infty 1/n = \infty.$$

Since $X_1, X_2, \ldots$ are i.i.d., the events $A_1, A_2, \ldots$ are independent. Thus by Lemma 1,

$$\mathbb{P}[|c_n X_n| \geq \varepsilon_0 \text{ i.o.}] = 1,$$

which proves Proposition 2. □