ECE 534 RANDOM PROCESSES **FALL 2015**
PROBLEM SET 2 **Due beginning of class, Thu, Sep 24**

2. Expectations and Limits

Assigned reading: Chapters 1 and 2 of the ECE 534 course notes. Be sure to at least skim through the problems at the end of the chapters.

Problems to be handed in:

1 The joys of symmetry

A real-valued random variable X is called *symmetric* if X and $-X$ have the same distribution, i.e., if the cdf's F_X and F_{-X} are equal.

(a) Let $X_1, \ldots, X_n$ be n independent Rademacher random variables, i.e., $\mathrm{P}[X_i = \pm 1] = 1/2$. Consider the random variable $Z = r_1 X_1 + \ldots + r_n X_n$, where $r_1, \ldots, r_n$ are arbitrary real constants. Prove that it is symmetric, and that

$$\mathbb{E}[Z^4] \le 3(\mathbb{E}[Z^2])^2.$$

(b) Let X be a symmetric random variable, and let S be a Rademacher random variable independent of X. Prove that X and $S|X|$ have the same distribution.

2 Indicator functions for fun and profit

In probability theory, one gets a great deal of mileage out of the fact that probabilities are expectations of indicator functions. In this problem, you will see how this simple observation can lead to a variety of useful inequalities:

(a) Let X be a nonnegative-valued random variable. Prove that, for any $t > 0$,

$$\mathbb{P}[X > t] \le \frac{\mathbb{E}[X \wedge t]}{t},$$

where $a \wedge b := \min(a, b)$. (Note that this improves upon Markov's inequality.)

(b) Again, let X be a nonnegative-valued random variable. Prove that, for any $t > 0$,

$$\mathbb{P}[X > t] \le \mathbb{E}[e^X]e^{-t}.$$

(c) Let X be a random variable taking values in the interval $[0, 1]$. Prove that

$$\mathbb{E}X \le t + \mathbb{P}[X > t], \qquad \forall t > 0.$$

(d) Let $A_1, \ldots, A_n$ be a collection of events, such that $\max_{1 \le i \le n} \mathbb{P}[A_i] > 0$. Prove that

$$\mathbb{P}\left[\bigcup_{i=1}^n A_i\right] \ge \frac{\sum_{i=1}^n \sum_{j=1}^n \mathbb{P}[A_i]\mathbb{P}[A_j]}{\sum_{i=1}^n \sum_{j=1}^n \mathbb{P}[A_i \cap A_j]}.$$

Hint: Let $B = \cup_i A_i$, and work with the identity

$$\sum_{i=1}^n \mathbf{1}_{A_i} = \mathbf{1}_B \sum_{i=1}^n \mathbf{1}_{A_i}.$$

3 Lower bounds for lower tails

(a) Let X be a nonnegative-valued random variable, such that $0 < \mathbb{E}X < \infty$. Prove the following inequality: for every $0 \le r \le 1$,

$$\mathbb{P}\left[X \ge r\,\mathbb{E}X\right] \ge (1-r)^2 \frac{(\mathbb{E}X)^2}{\mathbb{E}[X^2]}.$$

Hint: Split the expectation $\mathbb{E}[X]$ into two parts using the indicator $\mathbf{1}\{X \ge r\,\mathbb{E}X\}$.

(b) Now use this inequality to prove the following statement: If $X_1, \ldots, X_n$ are independent symmetric random variables, then, for any $r \in (0,1)$,

$$\mathbb{P}\left[\left(\sum_{i=1}^n X_i\right)^2 \ge r \sum_{i=1}^n X_i^2\right] \ge \frac{(1-r)^2}{3}.$$

Hint: Use the fact that, for each i, X_i has the same distribution as $S_i|X_i|$, where $S_1, \ldots, S_n$ are i.i.d. Rademacher random variables that are also independent of $X_1, \ldots, X_n$.

4 The importance of lowered expectations

Let $X, X_1, X_2, \ldots$ be a sequence of random variables on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Prove that $X_n \xrightarrow{P} X$ if and only if

$$\lim_{n\to\infty} \mathbb{E}\left[|X_n - X| \wedge 1\right] = 0.$$

5 Probably, but not surely

Let $X_1, X_2, \ldots$ be independent and identically distributed (i.i.d.) random variables, such that $\mathbb{P}[|X_n| > t] > 0$ for all $t > 0$. Prove that one can choose some constants $c_1, c_2, \ldots$, such that the sequence $c_n X_n$ converges to 0 in probability, but not almost surely.

Hint: Use the Borel–Cantelli lemma.