Homework 1: Solutions

September 16, 2015

1 Symmetric differences

A moment of reflection shows that

$$\mathbf{1}_{A\triangle B} = |\mathbf{1}_A - \mathbf{1}_B|\,.$$

Therefore,

$$\begin{aligned}
\mathbb{P}[A\triangle B] &= \mathbb{E}[\mathbf{1}_{A\triangle B}] \\
&= \mathbb{E}\,|\mathbf{1}_A - \mathbf{1}_C| \\
&\leq \mathbb{E}\,|\mathbf{1}_A - \mathbf{1}_C| + \mathbb{E}\,|\mathbf{1}_B - \mathbf{1}_C| \\
&= \mathbb{P}[A\triangle C] + \mathbb{P}[B\triangle C],
\end{aligned}$$

where in the third step we have used the triangle inequality.

2 Partitions and independence

Proof by contradiction. Suppose events G and H are mutually exclusive. Then $H \subseteq G^c = F$. This means that $\mathbb{P}[F|H] = 1$. By the assumption that F and H are independent, we have $\mathbb{P}[F] = \mathbb{P}[F|H] = 1$. By the assumption that F and G form a partition of Ω, we have $\mathbb{P}[G] = 0$, which contradicts the assumption that $\mathbb{P}[G] > 0$.

3 Random sets

(a) From the law of total probability and the assumption that A and B are independent and uniformly distributed over the power set of $\{1, \ldots, n\}$,

$$\begin{aligned}
\mathbb{P}[A \subseteq B] &= \sum_{s\subseteq\{1,\ldots,n\}} \mathbb{P}[B = s]\mathbb{P}[A \subseteq s | B = s] = \sum_{s\subseteq\{1,\ldots,n\}} \mathbb{P}[B = s]\mathbb{P}[A \subseteq s] \\
&= \sum_{s\subseteq\{1,\ldots,n\}} \frac{1}{2^n}\frac{2^{|s|}}{2^n} = \sum_{|s|=0}^{n} \frac{\binom{n}{|s|}}{2^n}\frac{2^{|s|}}{2^n} = \frac{1}{4^n}\sum_{|s|=0}^{n}\binom{n}{|s|}2^{|s|}1^{n-|s|} = \frac{1}{4^n}(2+1)^n \\
&= \left(\frac{3}{4}\right)^n.
\end{aligned}$$

(b) By the same token,

$$\begin{aligned}\mathbb{P}[A \cap B = \varnothing] &= \sum_{s \subseteq \{1,\ldots,n\}} \mathbb{P}[B = s]\mathbb{P}[A \subseteq \Omega \setminus s | B = s] = \sum_{s \subseteq \{1,\ldots,n\}} \mathbb{P}[B = s]\mathbb{P}[A \subseteq \Omega \setminus s] \\ &= \sum_{s \subseteq \{1,\ldots,n\}} \frac{1}{2^n}\frac{2^{n-|s|}}{2^n} = \sum_{|s|=0}^{n} \frac{\binom{n}{|s|}}{2^n}\frac{2^{n-|s|}}{2^n} = \frac{1}{4^n}\sum_{|s|=0}^{n}\binom{n}{|s|}2^{n-|s|}1^{|s|} = \frac{1}{4^n}(2+1)^n \\ &= \left(\frac{3}{4}\right)^n.\end{aligned}$$

4 Random graphs

(a) For a graph with n vertices, there are $\binom{n}{2}$ pairs of vertices. Each pair of vertices is either connected by an edge or not. There are $2^{\binom{n}{2}}$ different graphs.

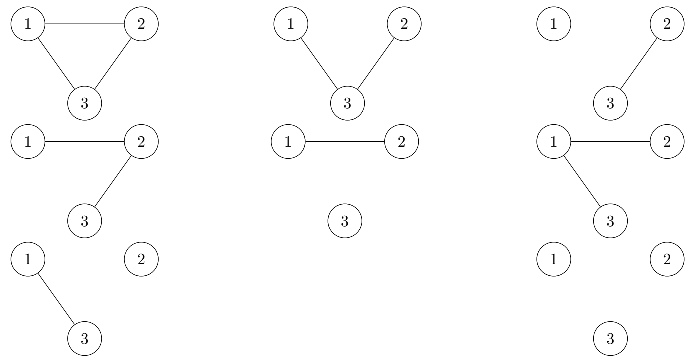

Figure 1: Sample space for $n = 3$ (from Joao Paulo's homework).

(b) Since each pair of vertices are connected by an edge independently of other pairs, the number of edges is a sum of $\binom{n}{2}$ independent Bernoulli random variables with bias p, namely, a Binomial$(\binom{n}{2}, p)$ random variable. Thus

$$\mathbb{P}[\mathbb{G}(n,p) \text{ has } k \text{ edges}] = \binom{m}{k}p^k(1-p)^{m-k}, \quad \text{where } m = \binom{n}{2}.$$

(c) Let $\mathcal{C}$ be the collection of the triples of vertices. Then, by definition of T and the linearity of expectation,

$$\mathbb{E}[T] = \mathbb{E}\left[\sum_{c \in \mathcal{C}} \mathbf{1}\{c \text{ is a triangle}\}\right] = \sum_{c \in \mathcal{C}} \mathbb{E}\left[\mathbf{1}\{c \text{ is a triangle}\}\right] = \sum_{c \in \mathcal{C}} \mathbb{P}[c \text{ is a triangle}] = \binom{n}{3}p^3.$$

Note that the random variables $\mathbf{1}\{c \text{ is a triangle}\}$ for different c are not necessarily independent, but the linearity of expectation does not rely on the independence among the random variables.

(d) Let $\mathcal{V}$ be the set of vertices. Then, by definition of S and the linearity of expectation,

$$\mathbb{E}[S] = \mathbb{E}\left[\sum_{v \in \mathcal{V}} \mathbf{1}\{v \text{ is isolated}\}\right] = \sum_{v \in \mathcal{V}} \mathbb{E}\left[\mathbf{1}\{v \text{ is isolated}\}\right] = \sum_{v \in \mathcal{V}} \mathbb{P}[v \text{ is isolated}] = n(1-p)^{n-1}.$$

Note that the random variables $\mathbf{1}\{v \text{ is isolated}\}$ for different v are not necessarily independent.

5 Integer-valued random variables

(a) Since $p_n \geq p_{n-1}$, we have $q_n \geq 0$. It suffices to show $\sum_{n=1}^{\infty} q_n = 1$.

Let $r_n = \sum_{i=1}^{n} p_i$ and $s_n = \sum_{i=1}^{n} q_i$. Then, from the telescopic sum,

$$s_n = r_n - np_{n+1}.$$

From the fact that $q_n \geq 0$ and $r_n \leq 1$, we know that s_n is a nondecreasing sequence bounded from above, hence $\lim_{n\to\infty} s_n$ exists. Since $\lim_{n\to\infty} r_n = 1$, $\lim_{n\to\infty} np_{n+1}$ also exists. It suffices to show $\lim_{n\to\infty} np_{n+1} = 0$.

We prove this by contradiction. Suppose $\lim_{n\to\infty} np_{n+1} = c > 0$. Then there is some $N_0 \in \mathbb{N}$, such that $np_{n+1} > c/2$ for all $n \geq N_0$, which means $p_{n+1} > c/(2n)$ for all $n \geq N_0$. This implies that the sequence p_n is not summable, which contradicts the assumption that $\lim_{n\to\infty} r_n = 1$. Therefore, $\lim_{n\to\infty} np_{n+1} = 0$, and $\lim_{n\to\infty} s_n = 1$.

(b) We have proved that the sequence $n(p_n - p_{n+1})$ is summable, thus $n(p_n - p_{n+1})$ must decay faster than $1/n$, and $p_n - p_{n+1}$ must decay faster than $1/n^2$.

6 Random sign flips

(a) By the linearity of covariance with respect to each argument,

$$\begin{aligned}\mathrm{Cov}[Y_v, Y_w] &= \mathrm{Cov}\left[\sum_{i=1}^{n} v_i Z_i, \sum_{j=1}^{n} w_j Z_j\right] = \sum_{i=1}^{n} v_i \mathrm{Cov}\left[Z_i, \sum_{j=1}^{n} w_j Z_j\right] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} v_i w_j \mathrm{Cov}[Z_i, Z_j] = \sum_{i=1}^{n} v_i w_i,\end{aligned}$$

where the last step follows from the fact that $\mathrm{Cov}[Z_i, Z_i] = \mathrm{Var}[Z_i] = 1$, and $\mathrm{Cov}[Z_i, Z_j] = 0$ for $i \neq j$. In other words, $\mathrm{Cov}[Y_v, Y_w] = \langle v, w\rangle$, the inner product of the vectors v and w. Next, since Z_i's are i.i.d.,

$$\mathrm{Var}[Y_v] = \sum_{i=1}^{n} v_i^2 \mathrm{Var}[Z_i] = \sum_{i=1}^{n} v_i^2 = \|v\|_2^2, \quad \mathrm{Var}[Y_w] = \sum_{i=1}^{n} w_i^2 \mathrm{Var}[Z_i] = \sum_{i=1}^{n} w_i^2 = \|w\|_2^2.$$

Thus

$$\rho_{Y_v, Y_w} = \frac{\mathrm{Cov}[Y_v, Y_w]}{\sqrt{\mathrm{Var}[Y_v]\mathrm{Var}[Y_w]}} = \frac{\langle v, w\rangle}{\|v\|_2\|w\|_2}.$$

That is, the correlation coefficient $\rho_{Y_v Y_w}$ is equal to the cosine of the angle between the vectors v and w.

(b) Let $\bar{Z}_i = Z_i Z_i'$. Then

$$P_{\bar{Z}_i|Z_i=1} = P_{\bar{Z}_i|Z_i=-1} = U\{\pm 1\},$$

which means that $\bar{Z}_i$ and Z_i are independent and have the same distribution. Moreover, since both Z_i's and Z_i''s are i.i.d., $\bar{Z}_i$'s are also i.i.d. It follows that Y_v and Y_v' have the same characteristic function, as both of which are determined by the marginal distributions of Z_i's and $\bar{Z}_i$'s. Y_v and Y_v' thus also have the same cdf.

(c) Let $S_n = \sum_{i=1}^{n} \mathbf{1}\{Z_i = 1\}$. Then

$$\mathbb{E}[\cos(\pi(Z_1 + \ldots + Z_n))] = \mathbb{E}[\cos(\pi(2S_n - n))] = \mathbb{E}[\cos(-\pi n)] = (-1)^n.$$

7 Quarter-circular world

A character-building exercise in calculus.

(a)

$$f_{X,Y}(x,y) = \mathbf{1}\{x, y \geq 0, x^2 + y^2 \leq 1\} \frac{1}{\text{area of the wedge}} = \mathbf{1}\{x, y \geq 0, x^2 + y^2 \leq 1\} \frac{4}{\pi}$$

(b)

$$f_X(x) = \int_{\mathbb{R}} f_{X,Y}(x,y) \mathrm{d}y = \int_0^{\sqrt{1-x^2}} \frac{4}{\pi} \mathrm{d}y = \frac{4\sqrt{1-x^2}}{\pi}, \quad 0 \leq x \leq 1.$$

(c)

$$\mathbb{E}[\sqrt{X^2 + Y^2}] = \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{4}{\pi} \sqrt{x^2 + y^2} \, \mathrm{d}y \, \mathrm{d}x = \frac{2}{3}.$$

8 Nonlinear transformation of a random variable

(a) Let $Y = g(X)$. Then

$$F_Y(y) = \mathbb{P}[Y \leq y] = \mathbb{P}[\{x : g(x) \leq y\}] = \begin{cases} 0, & y < 1/2 \\ y, & 1/2 \leq y \leq 1 \\ 1, & y \geq 1 \end{cases}.$$

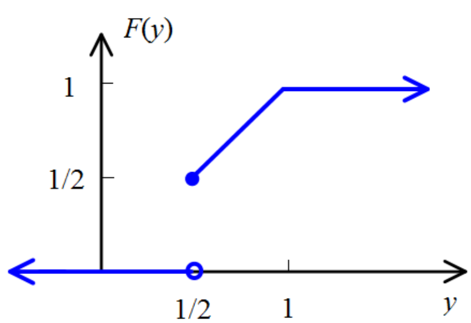

Figure 2: Sketch of $F_Y(y)$ (from Joao Paulo's homework).

The somewhat tricky part is to figure out what happens in the range $1/2 \leq y \leq 1$. We have

$$\begin{aligned} \mathbb{P}[Y \leq y] &= \mathbb{P}[1/2 \leq Y \leq y] \\ &= \mathbb{P}[Y = 1/2] + \mathbb{P}[1/2 < Y \leq y] \\ &= \frac{1}{2} + \mathbb{P}\left[\{-y \leq X < -1/2\} \cup \{1/2 < X \leq y\}\right] \\ &= \frac{1}{2} + 2 \cdot \frac{y - 1/2}{2} \\ &= y. \end{aligned}$$

(b)

$$f_Y(y) = \frac{\mathrm{d}F_Y(y)}{\mathrm{d}y} = \begin{cases} 0, & y < 1/2 \text{ or } y > 1 \\ \frac{1}{2}\delta(y - \frac{1}{2}), & y = 1/2, \\ 1, & 1/2 < y \leq 1 \end{cases}.$$

(c)

$$\mathbb{E}[Y] = \int_{\mathbb{R}} y f_Y(y)\,\mathrm{d}y = \frac{1}{4} + \int_{1/2}^{1} y\,\mathrm{d}y = \frac{5}{8}.$$

$$\mathbb{E}[Y^2] = \int_{\mathbb{R}} y^2 f_Y(y)\,\mathrm{d}y = \frac{1}{8} + \int_{1/2}^{1} y^2\,\mathrm{d}y = \frac{5}{12}.$$

$$\mathrm{Var}[Y] = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = \frac{5}{12} - \left(\frac{5}{8}\right)^2 = \frac{5}{192}.$$

(d) For $y = 1/2$,

$$f_{X|Y}(x|y) = \frac{f_X(x) P_{Y|X}(1/2|x)}{P_Y(1/2)} = \frac{\frac{1}{2}\mathbf{1}\{-1/2 \leq x \leq 1/2\}}{1/2} = \mathbf{1}\{-1/2 \leq x \leq 1/2\}.$$

Thus $\mathbb{E}[X|Y = 1/2] = 0$.

For $1/2 < y \leq 1$,

$$f_{X|Y}(x|y) = \frac{f_X(x) f_{Y|X}(y|x)}{f_Y(y)} = \frac{\frac{1}{2}\mathbf{1}\{-1 \leq x \leq 1\}\delta(y - |x|)}{1} = \begin{cases} \frac{1}{2}\delta(x - y), & x \geq 0 \\ \frac{1}{2}\delta(x + y), & x < 0 \end{cases}$$

Thus $\mathbb{E}[X|Y = y] = 0$ for $1/2 < y \leq 1$.

9 Soupe de Poisson

From the properties of Poisson random variables, $X_1 + X_2 \sim \mathrm{Poisson}(\lambda_1 + \lambda_2)$. For $k = 0, 1, \ldots,$

$$\begin{aligned} P_{X_1|X_1+X_2=n}(k) &= \frac{\mathbb{P}[X_1 = k, Z = n]}{\mathbb{P}[Z = n]} = \frac{\mathbb{P}[X_1 = k]\mathbb{P}[X_1 + X_2 = n | X_1 = k]}{\mathbb{P}[X_1 + X_2 = n]} \\ &= \frac{\mathbb{P}[X_1 = k]\mathbb{P}[X_2 = n - k | X_1 = k]}{\mathbb{P}[X_1 + X_2 = n]} = \frac{\mathbb{P}[X_1 = k]\mathbb{P}[X_2 = n - k]}{\mathbb{P}[X_1 + X_2 = n]} \\ &= \frac{e^{-\lambda_1}\frac{\lambda_1^k}{k!} e^{-\lambda_2}\frac{\lambda_2^{n-k}}{(n-k)!}}{e^{-(\lambda_1+\lambda_2)}\frac{(\lambda_1+\lambda_2)^n}{n!}} \\ &= \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1 + \lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1 + \lambda_2}\right)^{(n-k)}. \end{aligned}$$

Thus, conditional on $X_1 + X_2 = n$, X_1 is a $\mathrm{Binomial}(n, \lambda_1/(\lambda_1 + \lambda_2))$ random variable.

10 A doubly stochastic situation

(a) The z-transform of a nonnegative-integer-valued random variable X is

$$\Psi_X(z) = \sum_{i=0}^{\infty} P_X(i) z^i = \mathbb{E}[z^X].$$

By law of iterated expectation and the i.i.d. assumption of X_i's,

$$\Psi_Y(z) = \mathbb{E}[z^Y] = \mathbb{E}[\mathbb{E}[z^Y|N]] = \mathbb{E}[\mathbb{E}[z^{X_1+\ldots+X_N}|N]] = \mathbb{E}[\Psi_X(z)^N] = \Psi_N(\Psi_X(z))$$

(b) For $X \sim$ Bernoulli(p),

$$\Psi_X(z) = (1-p) + pz.$$

For $N \sim$ Poisson(λ),

$$\Psi_N(z) = \sum_{i=0}^{\infty} p_i z^i = e^{-\lambda} \sum_{i=0}^{\infty} \frac{(z\lambda)^i}{i!} = e^{-\lambda} e^{z\lambda} = e^{\lambda(z-1)}.$$

Thus

$$\Psi_Y(z) = e^{\lambda p(z-1)}$$