Vapnik–Chervonenkis classes

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A key result on the ERM algorithm, proved in the previous lecture, was that

$$P(\widehat{f}_n) \le L^*(\mathscr{F}) + 4\mathbb{E}R_n(\mathscr{F}(Z^n)) + \sqrt{\frac{2\log(1/\delta)}{n}}$$

with probability at least $1-\delta$. The quantity $R_n(\mathscr{F}(Z^n))$ appearing on the right-hand side of the above bound is the *Rademacher average* of the random set

$$\mathscr{F}(Z^n) = \left\{ \big(f(Z_1),\dots,f(Z_n)\big) : f \in \mathscr{F} \right\},$$

often referred to as the *projection* of $\mathscr{F}$ onto the sample Z^n. From this we see that a sufficient condition for the ERM algorithm to produce near-optimal hypotheses with high probability is that the expected Rademacher average $\mathbb{E}R_n(\mathscr{F}(Z^n)) = \tilde{O}(1/\sqrt{n})$, where the $\tilde{O}(\cdot)$ notation indicates that the bound holds up to polylogarithmic factors in n, i.e., there exists some positive polynomial function $p(\cdot)$ such that

$$\mathbb{E}R_n(\mathscr{F}(Z^n)) \le O\left(\sqrt{\frac{p(\log n)}{n}}\right).$$

Hence, a lot of effort in statistical learning theory is devoted to obtaining tight bounds on $\mathbb{E}R_n(\mathscr{F}(Z^n))$.

One way to guarantee an $\tilde{O}(1/\sqrt{n})$ bound on $\mathbb{E}R_n$ is if the "effective size" of the random set $\mathscr{F}(Z^n)$ is finite and grows polynomially with n. Then the Finite Class Lemma will tell us that

$$R_n(\mathscr{F}(Z^n)) = O\left(\sqrt{\frac{\log n}{n}}\right).$$

In general, a reasonable notion of "effective size" is captured by various *covering numbers* (see, e.g., the lecture notes by Mendelson [Men03] or the recent monograph by Talagrand [Tal05] for detailed expositions of the relevant theory). In this lecture, we will look at a simple combinatorial notion of effective size for classes of *binary-valued* functions. This particular notion has originated with the work of Vapnik and Chervonenkis [VC71], and was historically the first such notion to be introduced into statistical learning theory. It is now known as the *Vapnik–Chervonenkis* (or *VC*) *dimension*.

1 Vapnik–Chervonenkis dimension: definition

Definition 1. *Let $\mathscr{C}$ be a class of (measurable) subsets of some space Z. We say that a finite set $S = \{z_1,\dots,z_n\} \subset \mathsf{Z}$ is* shattered *by $\mathscr{C}$ if for every subset $S' \subseteq S$ there exists some $C \in \mathscr{C}$ such that $S' = S \cap C$.*

In other words, $S = \{z_1, \dots, z_n\}$ is shattered by $\mathscr{C}$ if for any binary n-tuple $b = (b_1, \dots, b_n) \in \{0,1\}^n$ there exists some $C \in \mathscr{C}$ such that

$$\left(\mathbf{1}_{\{z_1 \in C\}}, \dots, \mathbf{1}_{\{z_n \in C\}}\right) = b$$

or, equivalently, if

$$\left\{\left(\mathbf{1}_{\{z_1 \in C\}}, \dots, \mathbf{1}_{\{z_n \in C\}}\right) : C \in \mathscr{C}\right\} = \{0,1\}^n,$$

where we consider any two $C_1, C_2 \in \mathscr{C}$ as equivalent if $\mathbf{1}_{\{z_i \in C_1\}} = \mathbf{1}_{\{z_i \in C_2\}}$ for all $1 \le i \le n$.

Definition 2. *The* Vapnik–Chervonenkis dimension *(or the* VC dimension*) of* $\mathscr{C}$ *is*

$$V(\mathscr{C}) \triangleq \max\left\{n \in \mathbb{N} : \exists S \subset \mathsf{Z} \text{ such that } |S| = n \text{ and } S \text{ is shattered by } \mathscr{C}\right\}.$$

If $V(\mathscr{C}) < \infty$*, we say that* $\mathscr{C}$ *is a* VC class *(of sets).*

We can express the VC dimension in terms of *shatter coefficients* of $\mathscr{C}$: Let

$$\mathbb{S}_n(\mathscr{C}) \triangleq \sup_{S \subset \mathsf{Z}, |S|=n} |\{S \cap C : C \in \mathscr{C}\}|$$

denote the nth *shatter coefficient* of $\mathscr{C}$, where for each fixed S we consider any two $C_1, C_2 \in \mathscr{C}$ as equivalent if $S \cap C_1 = S \cap C_2$. Then

$$V(\mathscr{C}) = \max\left\{n \in \mathbb{N} : \mathbb{S}_n(\mathscr{C}) = 2^n\right\}.$$

The VC dimension $V(\mathscr{C})$ may be infinite, but it is always well-defined. This follows from the following lemma:

Lemma 1. *If* $\mathbb{S}_n(\mathscr{C}) < 2^n$*, then* $\mathbb{S}_m(\mathscr{C}) < 2^m$ *for all* $m > n$*.*

Proof. Suppose $\mathbb{S}_n(\mathscr{C}) < 2^n$. Consider any $m > n$. We will suppose that $\mathbb{S}_m(\mathscr{C}) = 2^m$ and derive a contradiction. By our assumption that $\mathbb{S}_m(\mathscr{F}) = 2^m$, there exists $S = \{z_1, \dots, z_m\} \in \mathsf{Z}^m$, such that for every binary n-tuple $b = (b_1, \dots, b_n)$ we can find some $C \in \mathscr{C}$ satisfying

$$\left(\mathbf{1}_{\{z_1 \in C\}}, \dots, \mathbf{1}_{\{z_n \in C\}}, \mathbf{1}_{\{z_{n+1} \in C\}}, \dots, \mathbf{1}_{\{z_m \in C\}}\right) = (b_1, \dots, b_n, 0, \dots, 0). \tag{1}$$

From (1) it immediately follows that

$$\left(\mathbf{1}_{\{z_1 \in C\}}, \dots, \mathbf{1}_{\{z_n \in C\}}\right) = (b_1, \dots, b_n). \tag{2}$$

Since $b = (b_1, \dots, b_n)$ was arbitrary, we see from (2) that $\mathbb{S}_n(\mathscr{C}) = 2^n$. This contradicts our assumption that $\mathbb{S}_n(\mathscr{C}) < 2^n$, so we conclude that $\mathbb{S}_m(\mathscr{C}) < 2^m$ whenever $m > n$ and $\mathbb{S}_n(\mathscr{F}) < 2^n$. $\square$

There is a one-to-one correspondence between binary-valued functions $f : \mathsf{Z} \to \{0,1\}$ and subsets of Z:

$$\forall f : \mathsf{Z} \to \{0,1\} \text{ let } C_f \triangleq \{z : f(z) = 1\}$$
$$\forall C \subseteq \mathsf{Z} \text{ let } f_C \triangleq \mathbf{1}_{\{C\}}.$$

Thus, we can extend the concept of shattering, as well as the definition of the VC dimension, to any class $\mathscr{F}$ of functions $f : \mathsf{Z} \to \{0,1\}$:

Definition 3. *Let $\mathscr{F}$ be a class of functions $f : \mathsf{Z} \to \{0,1\}$. We say that a finite set $S = \{z_1, \dots, z_n\} \subset \mathsf{Z}$ is shattered by $\mathscr{F}$ if it is shattered by the class*

$$\mathscr{C}_{\mathscr{F}} \triangleq \left\{ \mathbf{1}_{\{f=1\}} : f \in \mathscr{F} \right\},$$

where $\mathbf{1}_{\{f=1\}}$ is the indicator function of the set $C_f \triangleq \{z \in \mathsf{Z} : f(z) = 1\}$. The nth shatter coefficient of $\mathscr{F}$ is $\mathbb{S}_n(\mathscr{F}) = \mathbb{S}_n(\mathscr{C}_{\mathscr{F}})$, and the VC dimension of $\mathscr{F}$ is defined as $V(\mathscr{F}) = V(\mathscr{C}_{\mathscr{F}})$.

In light of these definitions, we can equivalently speak of the VC dimension of a class of sets or a class of binary-valued functions.

2 Examples of Vapnik–Chervonenkis classes

2.1 Semi-infinite intervals

Let $\mathsf{Z} = \mathbb{R}$ and take $\mathscr{C}$ to be the class of all intervals of the form $(-\infty, t]$ as t varies over $\mathbb{R}$. We will prove that $V(\mathscr{C}) = 1$. In view of Lemma 1, it suffices to show that (1) any one-point set $S = \{a\}$ is shattered by $\mathscr{C}$, and (2) no two-point set $S = \{a, b\}$ is shattered by $\mathscr{C}$.

Given $S = \{a\}$, choose any $t_1 < a$ and $t_2 > a$. Then $(-\infty, t_1] \cap S = \emptyset$ and $(-\infty, t_2] \cap S = S$. Thus, S is shattered by $\mathscr{C}$. This holds for every one-point set S, and therefore we have proved (1). To prove (2), let $S = \{a, b\}$ and suppose, without loss of generality, that $a < b$. Then there exists no $t \in \mathbb{R}$ such that $(-\infty, t] \cap S = \{b\}$. This follows from the fact that if $b \in (-\infty, t] \cap S$, then $t \ge b$. Since $b > a$, we must have $t > a$, so that $a \in (-\infty, t] \cap S$ as well. Since a and b are arbitrary, we see that no two-point subset of $\mathbb{R}$ can be shattered by $\mathscr{C}$.

2.2 Closed intervals

Again, let $\mathsf{Z} = \mathbb{R}$ and take $\mathscr{C}$ to be the class of all intervals of the form $[s, t]$ for all $s, t \in \mathbb{R}$. Then $V(\mathscr{C}) = 2$. To see this, we will show that (1) any two point set $S = \{a, b\}$ can be shattered by $\mathscr{C}$ and that (2) no three-point set $S = \{a, b, c\}$ can be shattered by $\mathscr{C}$.

For (1), let $S = \{a, b\}$ and suppose, without loss of generality, that $a < b$. Choose four points $t_1, t_2, t_3, t_4 \in \mathbb{R}$ such that $t_1 < t_2 < a < t_3 < b < t_4$. There are four subsets of S: $\emptyset$, $\{a\}$, $\{b\}$, and $\{a, b\} = S$. Then

$$[t_1, t_2] \cap S = \emptyset, \quad [t_2, t_3] \cap S = \{a\}, \quad [t_3, t_4] \cap S = \{b\}, \quad [t_1, t_4] \cap S = S.$$

Hence, S is shattered by $\mathscr{C}$. This holds for every two-point set in $\mathbb{R}$, which proves (1). To prove (2), let $S = \{a, b, c\}$ be an arbitrary three-point set with $a < b < c$. Then the intersection of any $[t_1, t_2] \in \mathscr{C}$ with S containing a and c must necessarily contain b as well. This shows that no three-point set can be shattered by $\mathscr{C}$, so by Lemma 1 we conclude that $V(\mathscr{C}) = 2$.

2.3 Closed halfspaces

Let $\mathsf{Z} = \mathbb{R}^2$, and let $\mathscr{C}$ consist of all closed halfspaces, i.e., sets of the form

$$\{z = (z_1, z_2) \in \mathbb{R}^2 : w_1 z_1 + w_2 z_2 \ge b\}$$

for all choices of $w_1, w_2, b \in \mathbb{R}$ such that $(w_1, w_2) \neq (0, 0)$. Then $V(\mathscr{C}) = 3$.

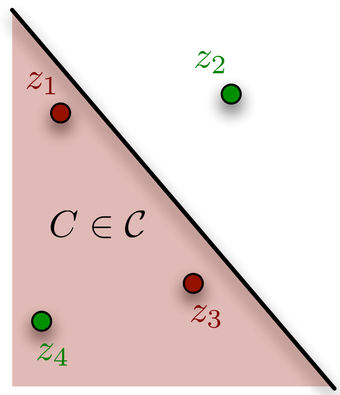

Figure 1: Impossibility of shattering an affinely independent four-point set in $\mathbb{R}^2$ by closed halfspaces.

To see that $\mathbb{S}_3(\mathscr{C}) = 2^3 = 8$, it suffices to consider any set $S = \{z_1, z_2, z_3\}$ of three *non-collinear* points. Then it is not hard to see that for any $S' \subseteq S$ it is possible to choose a closed halfspace $C \in \mathscr{C}$ that would contain S', but not S. To see that $\mathbb{S}_4(\mathscr{C}) < 2^4$, we must look at all four-point sets $S = \{z_1, z_2, z_3, z_4\}$. There are two cases to consider:

1. One point in S lies in the convex hull of the other three. Without loss of generality, let's suppose that $z_1 \in \operatorname{conv}(S')$ with $S' = \{z_2, z_3, z_4\}$. Then there is no $C \in \mathscr{C}$ such that $C \cap S = S'$. The reason for this is that every $C \in \mathscr{C}$ is a convex set. Hence, if $S' \subset C$, then any point in $\operatorname{conv}(S')$ is contained in C as well.

2. No point in S is in the convex hull of the remaining points. This case, when S is an *affinely independent set*, is shown in Figure 1. Let us partition S into two disjoint subsets, S_1 and S_2, each consisting of "opposite" points. In the figure, $S_1 = \{z_1, z_3\}$ and $S_2 = \{z_2, z_4\}$. Then it is easy to see that there is no halfspace $\mathscr{C}$ whose boundary could separate S_1 from its complement S_2. This is, in fact, the (in)famous "XOR counterexample" of Minsky and Papert [MP69], which has demonstrated the impossibility of universal concept learning by one-layer perceptrons.

Since any four-point set in $\mathbb{R}^2$ falls under one of these two cases, we have shown that no such set can be shattered by $\mathscr{C}$. Hence, $V(\mathscr{C}) = 3$.

More generally, if $\mathsf{Z} = \mathbb{R}^d$ and $\mathscr{C}$ is the class of all closed halfspaces

$$\left\{ z \in \mathbb{R}^d : \sum_{j=1}^{d} w_j z_j \geq b \right\}$$

for all $w = (w_1, \ldots, w_d) \in \mathbb{R}^d$ such that at least one of the w_j's is nonzero and all $b \in \mathbb{R}$, then $V(\mathscr{C}) = d + 1$ [WD81]; we will see a proof of this fact shortly.

2.4 Axis-parallel rectangles

Let $\mathsf{Z} = \mathbb{R}^2$, and let $\mathscr{C}$ consist of all "axis-parallel" rectangles, i.e., sets of the form $C = [a_1, b_1] \times [a_2, b_2]$ for all $a_1, b_1, a_2, b_2 \in \mathbb{R}$. Then $V(\mathscr{C}) = 4$.

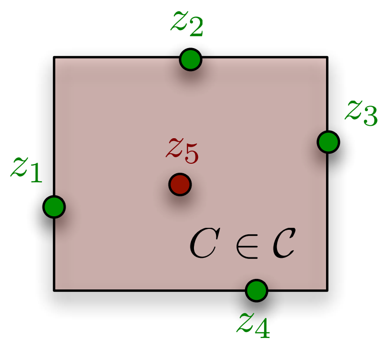

Figure 2: Impossibility of shattering a five-point set by axis-parallel rectangles.

First we exhibit a four-point set $S = \{z_1, z_2, z_3, z_4\}$ that is shattered by $\mathscr{C}$. It suffices to take $z_1 = (-2,-1)$, $z_2 = (1,-2)$, $z_3 = (2,1)$, $z_4 = (-1,2)$. To show that no five-point set is shattered by $\mathscr{C}$, consider an arbitrary $S = \{z_1, z_2, z_3, z_4, z_5\}$. Of these, pick any one point with the smallest first coordinate and any one point with the largest first coordinate, and likewise for the second coordinate (refer to Figure 2), for a total of at most four. Let S' denote the set consisting of these points; in Figure 2, $S' = \{z_1, z_2, z_3, z_4\}$. Then it is easy to see that any $C \in \mathscr{C}$ that contains the points in S' must contain all the points in $S \backslash S'$ as well. Hence, no five-point set in $\mathbb{R}^2$ can be shattered by $\mathscr{C}$, so $V(\mathscr{C}) = 5$.

The same argument also works for axis-parallel rectangles in $\mathbb{R}^d$, i.e., all sets of the form $C = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_d, b_d]$, leading to the conclusion that the VC dimension of the set of all axis-parallel rectangles in $\mathbb{R}^d$ is equal to $2d$.

2.5 Sets determined by finite-dimensional function spaces

The following result is due to Dudley [Dud78]. Let Z be arbitrary, and let $\mathscr{G}$ be an m-dimensional linear space of functions $g : \mathsf{Z} \to \mathbb{R}$, which means that each $g \in \mathscr{G}$ has a unique representation of the form

$$g = \sum_{j=1}^{m} c_j \psi_j,$$

where $\psi_1, \ldots, \psi_m : \mathsf{Z} \to \mathbb{R}$ form a fixed linearly independent set and $c_1, \ldots, c_m$ are real coefficients. Consider the class

$$\mathscr{C} = \Big\{ \{z \in \mathsf{Z} : g(z) \ge 0\} : g \in \mathscr{G} \Big\}.$$

Then $V(\mathscr{C}) \le m$.

To prove this, we need to show that no set of $m+1$ points in Z can be shattered by $\mathscr{C}$. To that end, let us fix $m+1$ arbitrary points $z_1, \ldots, z_{m+1} \in \mathsf{Z}$ and consider the mapping $L : \mathscr{G} \to \mathbb{R}^{m+1}$ defined by

$$L(g) \triangleq \big(g(z_1), \ldots, g(z_{m+1})\big).$$

It is easy to see that because $\mathscr{G}$ is a linear space, L is a linear mapping, i.e., for any $g_1, g_2 \in \mathscr{G}$ and any $c_1, c_2 \in \mathbb{R}$ we have $L(c_1 g_1 + c_2 g_2) = c_1 L(g_1) + c_2 L(g_2)$. Since $\dim \mathscr{G} = m$, the image of $\mathscr{G}$ under L, i.e., the set

$$L(\mathscr{G}) = \big\{ (g(z_1), \ldots, g(z_{m+1})) \in \mathbb{R}^{m+1} : g \in \mathscr{G} \big\},$$

is a linear subspace of $\mathbb{R}^{m+1}$ of dimension at most m. This means that there exists some nonzero vector $v = (v_1, \dots, v_{m+1}) \in \mathbb{R}^{m+1}$ orthogonal to $L(\mathscr{G})$, i.e., for every $g \in \mathscr{G}$

$$v_1 g(z_1) + \ldots + v_{m+1} g(z_{m+1}) = 0. \tag{3}$$

Without loss of generality, we may assume that at least one component of v is strictly negative (otherwise we can take $-v$ instead of v and still get (3)). Hence, we can rearrange the equality in (3) as

$$\sum_{i: v_i \ge 0} v_i g(z_i) = -\sum_{i: v_i < 0} v_i g(z_i), \qquad \forall g \in \mathscr{G}. \tag{4}$$

Now let us suppose that $\mathbb{S}_{m+1}(\mathscr{C}) = 2^{m+1}$ and derive a contradiction. Consider a binary $(m+1)$-tuple $b = (b_1, \dots, b_{m+1}) \in \{0,1\}^{m+1}$, where $b_j = 1$ if and only if $v_j \ge 0$, and 0 otherwise. Since we assumed that $\mathbb{S}_{m+1}(\mathscr{C}) = 2^{m+1}$, there exists some $g \in \mathscr{G}$ such that

$$\left(\mathbf{1}_{\{g(z_1) \ge 0\}}, \dots, \mathbf{1}_{\{g(z_{m+1}) \ge 0\}}\right) = b.$$

By our definition of b, this means that the left-hand side of (4) is nonnegative, while the right-hand side is negative, which is a contradiction. Hence, $\mathbb{S}_{m+1}(\mathscr{C}) < 2^{m+1}$, so $V(\mathscr{C}) \le m$.

This result can be used to bound the VC dimension of many classes of sets:

- Let $\mathscr{C}$ be the class of all closed halfspaces in $\mathbb{R}^d$. Then any $C \in \mathscr{C}$ can be represented in the form $C = \{z : g(z) \ge 0\}$ for $g(z) = \langle w, z \rangle - b$ with some nonzero $w \in \mathbb{R}^d$ and $b \in \mathbb{R}$. The set $\mathscr{G}$ of all such affine functions on $\mathbb{R}^d$ is a linear space of dimension $d+1$, so by the above result we have $V(\mathscr{C}) \le d+1$. In fact, we know that this holds with equality [WD81]. This can also be seen from the following result, due to Cover [Cov65]: Let $\mathscr{G}$ be the linear space of functions spanned by functions $\psi_1, \dots, \psi_m$, and let $\{z_1, \dots, z_n\} \subset \mathsf{Z}$ be such that the vectors $\Psi(z_i) = (\psi_1(z_i), \dots, \psi_m(z_i)), 1 \le i \le n$, form a linearly independent set. Then for the class of sets $\mathscr{C} = \{\{z : g(z) \ge 0\} : z \in \mathsf{Z}\}$ we have

  $$|C \cap \{z_1, \dots, z_n\} : C \in \mathscr{C}| = \sum_{i=0}^{m-1} \binom{n-1}{i}.$$

  The conditions needed for Cover's result are seen to hold for indicators of halfspaces, so letting $n = m = d+1$ we see that $\mathbb{S}_d(\mathscr{C}) = \sum_{i=0}^{d} \binom{d}{i} = 2^d$. Hence, $V(\mathscr{C}) = d+1$.

- Let $\mathscr{C}$ be the class of all closed balls in $\mathbb{R}^d$, i.e., sets of the form

  $$C = \left\{ z \in \mathbb{R}^d : \|z - x\|^2 \le r^2 \right\}$$

  where $x \in \mathbb{R}^d$ is the *center* of C and $r \in \mathbb{R}^+$ is its *radius*. Then we can write $C = \{z : g(z) \ge 0\}$, where

  $$g(z) = r^2 - \|z - x\|^2 = r^2 - \sum_{j=1}^{d} |z_j - x_j|^2. \tag{5}$$

  Expanding the second expression for g in (5), we get

  $$g(z) = r^2 - \sum_{j=1}^{d} x_j^2 + 2\sum_{j=1}^{d} x_j z_j - \sum_{j=1}^{d} z_j^2,$$

  which can be written in the form $g(z) = \sum_{k=1}^{d+2} c_k \psi_k(z)$, where $\psi_1(z) = 1$, $\psi_k(z) = z_k$ for $k = 2, \dots, d+1$, and $\psi_{d+2} = \sum_{j=1}^{d} z_j^2$. It can be shown that the functions $\{\psi_k\}_{k=1}^{d+2}$ are linearly independent. Hence, $V(\mathscr{C}) \le d+2$. This bound, however, is not tight; as shown by Dudley [Dud79], the class of closed balls in $\mathbb{R}^d$ has VC dimension $d+1$.

2.6 VC dimension vs. number of parameters

Looking back at all these examples, one may get the impression that the VC dimension of a set of binary-valued functions is just the number of parameters. This is not the case. Consider the following one-parameter family of functions:

$$g_\theta(z) \triangleq \sin(\theta z), \qquad \theta \in \mathbb{R}.$$

However, the class of sets

$$\mathscr{C} = \Big\{\{z \in \mathbb{R} : g_\theta(z) \ge 0\} : \theta \in \mathbb{R}\Big\}$$

has infinite VC dimension. Indeed, for any n, any collection of numbers $z_1, \ldots, z_n \in \mathbb{R}$, and any binary string $b \in \{0,1\}^n$, one can always find some $\theta \in \mathbb{R}$ such that

$$\operatorname{sgn}(\sin(\theta z_i)) = \begin{cases} +1, & \text{if } b_i = 1 \\ -1, & \text{if } b_i = 0 \end{cases}.$$

3 Growth of shatter coefficients: the Sauer–Shelah lemma

The importance of VC classes in learning theory arises from the fact that, as n tends to infinity, the fraction of subsets of any $\{z_1, \ldots, z_n\} \subset \mathsf{Z}$ that are shattered by a given VC class $\mathscr{C}$ tends to zero. We will prove this fact in this section by deriving a sharp bound on the shatter coefficients $\mathbb{S}_n(\mathscr{C})$ of a VC class $\mathscr{C}$. This bound have been (re)discovered at least three times, first in a weak form by Vapnik and Chervonenkis [VC71] in 1971, then independently and in different contexts by Sauer [Sau72] and Shelah [She72] in 1972. In strict accordance with Stigler's law of eponymy[1], it is known in the statistical learning literature as the *Sauer–Shelah lemma*.

Before we state and prove this result, we will collect some preliminaries and set up some notation. Given integers $n, d \ge 1$, let

$$\phi(n,d) \triangleq \begin{cases} \sum_{i=0}^{d} \binom{n}{i}, & \text{if } n > d \\ 2^n, & \text{if } n \le d \end{cases}$$

If we adopt the convention that $\binom{n}{i} = 0$ for $i > n$, we can write

$$\phi(n,d) = \sum_{i=0}^{d} \binom{n}{i}$$

for all $n, d \ge 1$. We will find the following recursive relation useful:

Lemma 2.

$$\phi(n,d) = \phi(n-1,d) + \phi(n-1,d-1).$$

Proof. We have

$$\binom{n-1}{i-1} + \binom{n-1}{i} = \frac{(n-1)!}{(i-1)!(n-i)!} + \frac{(n-1)!}{i!(n-i-1)!}.$$

[1] "No scientific discovery is named after its original discoverer" (http://en.wikipedia.org/wiki/Stigler's_law_of_eponymy)

Multiplying both sides by $i!(n-i)!$, we obtain

$$i!(n-i)!\left[\binom{n-1}{i-1}+\binom{n-1}{i}\right]=i(n-1)!+(n-i)(n-1)!=n!$$

Hence,

$$\binom{n-1}{i-1}+\binom{n-1}{i}=\frac{n!}{i!(n-i)!}=\binom{n}{i}. \tag{6}$$

Using the definition of $\phi(n,d)$, as well as (6), we get

$$\phi(n,d)=\sum_{i=0}^{d}\binom{n}{i}=1+\sum_{i=1}^{d}\binom{n}{i}=\underbrace{1+\sum_{i=1}^{d}\binom{n-1}{i}}_{=\phi(n-1,d)}+\underbrace{\sum_{i=1}^{d}\binom{n-1}{i-1}}_{=\phi(n-1,d-1)}$$

and the lemma is proved. □

Now for the actual result:

Theorem 1 (Sauer–Shelah lemma). *Let $\mathscr{C}$ be a class of subsets of some space Z with $V(\mathscr{C})=d<\infty$. Then for all n,*

$$\mathbb{S}_n(\mathscr{C})\le\phi(n,d). \tag{7}$$

Proof. There are several different proofs in the literature; we will use an inductive argument following Blumer et al. [BEHW89].

We can assume, without loss of generality, that $n>d$, for otherwise $\mathbb{S}_n(\mathscr{C})=2^n=\phi(n,d)$. For an arbitrary finite set $S\subset\mathsf{Z}$, let

$$\mathbb{S}(S,\mathscr{C})\triangleq|\{S\cap C: C\in\mathscr{C}\}|,$$

where, as before, we count only the distinct sets of the form $S\cap C$. By definition, $\mathbb{S}_n(\mathscr{C})=\sup_{S:|S|=n}\mathbb{S}(S,\mathscr{C})$. Thus, it suffices the prove the following: For any $S\subset\mathsf{Z}$ with $|S|=n>d$, $\mathbb{S}(S,\mathscr{C})\le\phi(n,d)$.

For the purpose of computing $\mathbb{S}(S,\mathscr{C})$, any two $C_1,C_2\in\mathscr{C}$ such that $S\cap C_1=S\cap C_2$ are deemed equivalent. Hence, let

$$\mathscr{A}\triangleq\{A\subseteq S: A=S\cap C \text{ for some } C\in\mathscr{C}\}.$$

Then we may write

$$\mathbb{S}(S,\mathscr{C})=|\{S\cap C: C\in\mathscr{C}\}|=|\{A\subseteq S: A=S\cap C \text{ for some } C\in\mathscr{C}\}|=|\mathscr{A}|.$$

Moreover, it is easy to see that $V(\mathscr{A})\le V(\mathscr{C})=d$.

Thus, the desired result is equivalent to saying that if $\mathscr{A}$ is a collection of subsets of an n-element set S (which we may, without loss of generality, take to be $[n]\triangleq\{1,\dots,n\}$) with $V(\mathscr{A})\le d<n$, then $|\mathscr{A}|\le\phi(n,d)$. We will prove this statement by "double induction" on n and d. First of all, the statement (7) holds for all $n\ge1$ and $d=0$. Indeed, if $V(\mathscr{A})=0$, then $|\mathscr{A}|=1\le2^n$. Now assume that (7) holds for all n and all $\mathscr{A}$ with $V(\mathscr{A})\le d-1$, and for all integers up to $n-1$ and all $\mathscr{A}$ with $V(\mathscr{A})\le d$. Now let $S=[n]$, and let $\mathscr{A}$ be a collection of subsets of $[n]$ with $V(\mathscr{A})=d<n$. We will show that $|\mathscr{A}|\le\phi(n,d)$.

To prove this claim, let us choose an arbitrary $i \in S$ and define

$$\begin{aligned} \mathscr{A} \setminus i &\triangleq \{A \setminus \{i\} : A \in \mathscr{A}\} \\ \mathscr{A}_i &\triangleq \{A \in \mathscr{A} : i \notin A, A \cup \{i\} \in \mathscr{A}\} \end{aligned}$$

Observe that both $\mathscr{A} \setminus i$ and $\mathscr{A}_i$ are classes of subsets of $S \setminus \{i\}$. Moreover, since A and $A \cup \{i\}$ map to the same element of $\mathscr{A} \setminus i$, while $|\mathscr{A}_i|$ is the number of pairs of sets in $\mathscr{A}$ that map into the same set in $\mathscr{A} \setminus i$, we have

$$|\mathscr{A}| = |\mathscr{A} \setminus i| + |\mathscr{A}_i|. \tag{8}$$

Since $\mathscr{A} \setminus i \subseteq \mathscr{A}$, we have $V(\mathscr{A} \setminus i) \le V(\mathscr{A}) \le d$. Also, every set in $\mathscr{A} \setminus i$ is a subset of $S \setminus \{i\}$, which has cardinality $n-1$. Therefore, by the inductive hypothesis $|\mathscr{A} \setminus i| \le \phi(n-1, d)$. Next, we show that $V(\mathscr{A}_i) \le d-1$. Suppose, to the contrary, that $V(\mathscr{A}_i) = d$. Then there must exist some $T \subseteq S \setminus \{i\}$ with $|T| = d$ that is shattered by $\mathscr{A}_i$. But then $T \cup \{i\}$ is shattered by $\mathscr{A}$. To see this, given any $T' \subseteq T$ choose some $A \in \mathscr{A}_i$ such that $T \cap A = T'$ (this is possible since T is shattered by $\mathscr{A}_i$). But then $A \cup \{i\} \in \mathscr{A}$ (by definition of $\mathscr{A}_i$), and

$$(T \cup \{i\}) \cap (A \cup \{i\}) = (T \cap A) \cup \{i\} = T' \cup \{i\}.$$

Since this is possible for an arbitrary $T' \subseteq T$, we conclude that $T \cup \{i\}$ is shattered by $\mathscr{A}$. Now, since $T \subseteq S \setminus \{i\}$, we must have $i \neq T$, so $|T \cup \{i\}| = |T| + 1 = d+1$, which means that there exists a $(d+1)$-element subset of $S = [n]$ that is shattered by $\mathscr{A}$. But this contradicts our assumption that $V(\mathscr{A}) \le d$. Hence, $V(\mathscr{A}_i) \le d-1$. Since $\mathscr{A}_i$ is a collection of subsets of $S \setminus \{i\}$, we must have $|\mathscr{A}_i| \le \phi(n-1, d-1)$ by the inductive hypothesis. Hence, from (8) and from Lemma 2 we have

$$|\mathscr{A}| = |\mathscr{A} \setminus i| + |\mathscr{A}_i| \le \phi(n-1, d) + \phi(n-1, d-1) = \phi(n, d).$$

This completes the induction argument and proves (7). $\square$

Corollary 1. *If $\mathscr{C}$ is a collection of sets with $V(\mathscr{C}) \le d < \infty$, then*

$$\mathbb{S}_n(\mathscr{C}) \le (n+1)^d.$$

Moreover, if $n \ge d$, then

$$\mathbb{S}_n(\mathscr{C}) \le \left(\frac{en}{d}\right)^d,$$

where e is the base of the natural logarithm.

Proof. For the first bound, write

$$\phi(n, d) = \sum_{i=0}^{d} \binom{n}{i} = \sum_{i=1}^{d} \frac{n!}{i!(n-i)!} \le \sum_{i=1}^{d} \frac{n^i}{i!} \le \sum_{i=0}^{d} \frac{n^i d!}{i!(d-i)!} = \sum_{i=0}^{d} n^i \binom{d}{i} = (n+1)^d,$$

where the last step uses the binomial theorem. On the other hand, if $d/n \le 1$, then

$$\left(\frac{d}{n}\right)^d \phi(n, d) = \left(\frac{d}{n}\right)^d \sum_{i=0}^{d} \binom{n}{i} \le \sum_{i=1}^{d} \left(\frac{d}{n}\right)^i \binom{n}{i} \le \sum_{i=1}^{n} \left(\frac{d}{n}\right)^i \binom{n}{i} = \left(1 + \frac{d}{n}\right)^n \le e^d,$$

where we again used the binomial theorem. Dividing both sides by $(d/n)^d$, we get the second bound. $\square$

Let $\mathscr{C}$ be a VC class of subsets of some space Z. From the above corollary we see that

$$\limsup_{n\to\infty} \frac{\mathbb{S}_n(\mathscr{C})}{2^n} \le \lim_{n\to\infty} \frac{(n+1)^{V(\mathscr{C})}}{2^n} = 0.$$

In other words, as n becomes large, the fraction of subsets of an arbitrary n-element set $\{z_1, \dots, z_n\} \subset \mathsf{Z}$ that are shattered by $\mathscr{C}$ becomes negligible. Moreover, combining the bounds of the corollary with the Finite Class Lemma for Rademacher averages, we get the following:

Theorem 2. *Let $\mathscr{F}$ be a VC class of binary-valued functions $f : \mathsf{Z} \to \{0,1\}$ on some space Z. Let Z^n be an i.i.d. sample of size n drawn according to an arbitrary probability distribution $P \in \mathscr{P}(\mathsf{Z})$. Then*

$$\mathbb{E} R_n(\mathscr{F}(Z^n)) \le 2\sqrt{\frac{V(\mathscr{F})\log(n+1)}{n}}.$$

A more refined *chaining technique* [Dud78] can be used to remove the logarithm in the above bound:

Theorem 3. *There exists an absolute constant $C > 0$, such that under the conditions of the preceding theorem*

$$\mathbb{E} R_n(\mathscr{F}(Z^n)) \le C\sqrt{\frac{V(\mathscr{F})}{n}}.$$

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