ECE 534 RANDOM PROCESSES **FALL 2012**
SOLUTIONS FOR PROBLEM SET 6

1 (7.1) Calculus for a simple Gaussian random process

(a): In order to verify this directly, we have to use the definition. We first calculate $\frac{X_s - X_t}{s-t}$ which is equal to $B + C(t+s)$. Now, we have to show that the fraction $\frac{X_s - X_t}{s-t}$ converges to $B + 2Ct$ as $s \to t$ in mean square sense:

$$\begin{aligned}
\lim_{s \to t} \mathbb{E}\left[\left(\frac{X_s - X_t}{s-t} - (B+2Ct)\right)^2\right] &= \lim_{s \to t} \mathbb{E}\left[(C(s-t))^2\right] \\
&= \lim_{s \to t}(s-t)^2 \mathbb{E}\left[C^2\right] = \lim_{s \to t}(s-t)^2 = 0
\end{aligned}$$

Therefore, X is m.s. differentiable.

(b): Since $X_t = A + Bt + Ct^2$ is a linear combination of three independant Gaussian random variables, X_t is a Gaussian random variable itself. On the other hand, any linear combination of $X_t; t \in \mathbb{T}$ can be expressed as linear combination of Gaussian random variables A, B, C thus is a Gaussian itslef. This argument shows that X_t's are jointly Gaussian for different values of t.

Therefore, $Y_t = \int_0^t X_s ds$ which is a linear combination of jointly Gaussian random variables is a Gaussian for which we only need its mean and variance. We have:

$$\begin{aligned}
\mathbb{E}\left[\int_0^t X_s ds\right] &= \int_0^t \mathbb{E}\left[X_s\right] ds = 0 \\
\text{Var}\left[\int_0^t X_s ds\right] &= \mathbb{E}\left[\left(\int_0^t X_s ds\right)^2\right] - \mathbb{E}^2\left[\int_0^t X_s ds\right] \\
&= \mathbb{E}\left[\left(\int_0^t X_s ds\right)^2\right] \\
&= \iint_0^1 R_X(s,t) ds\ dt \\
&= \iint_0^1 \mathbb{E}\left[(A + Bs + Cs^2)(A + Bt + Ct^2)\right] \\
&= \iint_0^1 \left(\mathbb{E}\left[A^2\right] + st\mathbb{E}[B^2] + s^2t^2\mathbb{E}[C^2]\right) ds\ dt \\
&= \iint_0^1 \left(1 + st + s^2t^2\right) ds\ dt = \frac{49}{36}
\end{aligned}$$

Therefore, $\int_0^t X_s ds \sim \mathcal{N}(0, \frac{49}{36})$ and we have $\mathcal{P}\left(\int_0^t X_s ds \geq 1\right) = Q\left(\frac{1}{\sqrt{\frac{49}{36}}}\right) = Q(\frac{6}{7})$

2 (7.3) Properties of a binary valued process

(a) Yes, This is a markov process. Consider a sequence $0 < t_1 < t_2 < \cdots < t_n < t$, we have:

$$\begin{aligned}\mathbb{P}(Y_t = c | Y_{t_1} = c_1, Y_{t_2} = c_2, \cdots, Y_{t_n} = c_n) =& \mathbb{P}((-1)^{N_t} = c | Y_{t_1} = c_1, Y_{t_2} = c_2, \cdots, Y_{t_n} = c_n) \\ =& \mathbb{P}((-1)^{N_{t_n}}(-1)^{N_t - N_{t_n}} = c | Y_{t_1} = c_1, Y_{t_2} = c_2, \cdots, Y_{t_n} = c_n) \\ =& \mathbb{P}((-1)^{N_t - N_{t_n}} = \frac{c}{c_n} | Y_{t_1} = c_1, Y_{t_2} = c_2, \cdots, Y_{t_n} = c_n) \\ =& \mathbb{P}((-1)^{N_t - N_{t_n}} = \frac{c}{c_n})\end{aligned}$$

where the last equation holds because of the independant increment property of poisson process. For the transition probability matrix we have:

$$\begin{aligned} p_{i,j}(s,t) &= \begin{cases} P\{N_t - N_s \text{ is even}\} & i = j \\ P\{N_t - N_s \text{ is odd}\} & i \neq j \end{cases} \\ &= \begin{cases} \sum_{n=0}^{\infty} P\{N_t - N_s = 2n\} & i = j \\ \sum_{n=0}^{\infty} P\{N_t - N_s = 2n+1\} & i \neq j \end{cases} \\ &= \begin{cases} \sum_{n=0}^{\infty} e^{-\lambda(t-s)} \frac{(\lambda(t-s))^{2n}}{(2n)!} & i = j \\ \sum_{n=0}^{\infty} e^{-\lambda(t-s)} \frac{(\lambda(t-s))^{2n+1}}{(2n+1)!} & i \neq j \end{cases} \\ &= \begin{cases} e^{-\lambda(t-s)} \cosh(\lambda(t-s)) & i = j \\ e^{-\lambda(t-s)} \sinh(\lambda(t-s)) & i \neq j \end{cases} \\ &= \begin{cases} e^{-\lambda(t-s)} \frac{e^{\lambda(t-s)} + e^{-\lambda(t-s)}}{2} & i = j \\ e^{-\lambda(t-s)} \frac{e^{\lambda(t-s)} - e^{-\lambda(t-s)}}{2} & i \neq j \end{cases} \\ &= \begin{cases} \frac{1 + e^{-2\lambda(t-s)}}{2} & i = j \\ \frac{1 - e^{-2\lambda(t-s)}}{2} & i \neq j \end{cases} \end{aligned}$$

Having transition probability, one easily has:

$$Q = \begin{pmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{pmatrix}$$

(b) Yes. Y_t is mean square continuous. The easiest way to prove is to directly use the definition of m.s. continuity:

$$
\begin{aligned}
\lim_{t\to s}\mathbb{E}\left[(Y_t-Y_s)^2\right] &= \lim_{t\to s}\mathbb{E}\left[\left((-1)^{N_t}-(-1)^{N_s}\right)^2\right]\\
&= \lim_{t\to s}\mathbb{E}\left[\mathbb{E}\left[\left((-1)^{N_t}-(-1)^{N_s}\right)^2 \mid N_t-N_s\right]\right]\\
&= \lim_{t\to s}\mathbb{E}\left[2^2 I_{\{N_t-N_s \text{ is odd}\}}+0I_{\{N_t-N_s \text{ is even}\}}\right]\\
&= \lim_{t\to s}4P\{N_t-N_s \text{ is odd}\}\\
&= \lim_{t\to s}2-2e^{-2\lambda|t-s|}=0
\end{aligned}
$$

(c) No. Y_t is not differentiable in mean square sense. For this, we use the fact that a random process Y is differentiable in m.s. sense if and only if R_Y and its first-order partial derivatives exist and are continuous. Here, we have:

$$
\begin{aligned}
R_Y(s,t) =& \mathbb{E}\left[Y_sY_t\right]\\
=& \mathbb{E}\left[(-1)^{N_s}(-1)^{N_t}\right]\\
=& \mathbb{E}\left[I_{\{N_s-N_t \text{ is even}\}}-I_{\{N_s-N_t \text{ is odd}\}}\right]\\
=& P\{N_s-N_t \text{ is even}\}-P\{N_s-N_t \text{ is odd}\}\\
=& \frac{1+e^{-2\lambda|t-s|}}{2}-\frac{1-e^{-2\lambda|t-s|}}{2}\\
=& e^{-2\lambda|t-s|}
\end{aligned}
$$

R_Y is continuous but its derivatives are discontinuous around $t = s$. Therefore Y is not m.s. differentiable.

(d) Yes. This limit exists in the m.s. sense and is equal zero.

To see why, first note that since R_Y is continuous, the integral itself exists in the m.s. sense. Then, the limit of time average exists in the m.s. sense if $\mathbb{E}\left[\left(\frac{1}{T}\int_0^T y_t dt - 0\right)^2\right]$ can be made arbitrarily small as $T \to \infty$.

$$
\begin{aligned}
\mathbb{E}\left[\left(\frac{1}{T}\int_0^T y_t dt - 0\right)^2\right] =& \frac{1}{T^2}\mathbb{E}\left[\left(\int_0^T y_t dt\right)^2\right] \\
=& \frac{1}{T^2}\int_0^T\int_0^T R_Y(s,t)dsdt \\
=& \frac{1}{T^2}\int_0^T\int_0^T e^{-2\lambda|t-s|}dsdt \\
=& \frac{1}{T^2}\int_0^T\int_0^t e^{-2\lambda(t-s)}dsdt + \frac{1}{T^2}\int_0^T\int_0^s e^{-2\lambda(s-t)}dtds \\
=& \frac{2}{T^2}\int_0^T\int_0^t e^{-2\lambda(t-s)}dsdt \\
=& \frac{2}{T^2}\int_0^T \left[\frac{e^{-2\lambda(t-s)}}{2\lambda}\right]_0^t dt \\
=& \frac{2}{T^2}\int_0^T \frac{1-e^{-2\lambda t}}{2\lambda}dt \\
=& \frac{2}{T^2}\left[\frac{t}{2\lambda} - \frac{e^{-2\lambda t}}{-4\lambda^2}\right]_0^T \\
=& \frac{2}{T^2}\left[\frac{T}{2\lambda} + \frac{e^{-2\lambda T}-1}{4\lambda^2}\right] \\
=& \frac{1}{\lambda T} + \frac{e^{-2\lambda T}-1}{2\lambda^2 T^2}
\end{aligned}
$$

As we can see, $\lim_{T\to\infty} \frac{1}{\lambda T} + \frac{e^{-2\lambda T}-1}{2\lambda^2 T^2} = 0$ and our claim is justified.

3 (7.15) Integral of a Brownian bridge

(a) To see if B_t defined as $B_t = W_t - tW_1$ is integrable, we have to see whether its autocorrelation function is integrable and if it is finite.

$$
\begin{aligned}
\int_0^1\int_0^1 R_B(s,t)dsdt =& \int_0^1\int_0^t s(1-t)dsdt + \int_0^1\int_0^s t(1-s)dtds \\
=& 2\int_0^1\int_0^t s(1-t)dsdt \\
=& 2\int_0^1 \frac{t^2}{2}(1-t)dt \\
=& 2\left[\frac{t^3}{6} - \frac{t^4}{8}\right]_0^1 = \frac{1}{12}
\end{aligned}
$$

Since R_B is integrable on $[0,1]$, B_t is also integrable on $[0,1]$, so $X = \int_0^1 B_t dt$ is well defined in the m.s. sense.

(b) For the joint distribution of a jointly Gaussian random variable, we need to speciy means and correlations. We have:

$$\begin{aligned}
\mathbb{E}[X] =& \mathbb{E}\left[\int_0^1 W_t dt\right] - \mathbb{E}\left[W_1 \int_0^1 t\ dt\right] \\
=& \mathbb{E}\left[\int_0^1 W_t dt\right] - \frac{1}{2}\mathbb{E}\left[W_1\right] = 0
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(X) =& \mathbb{E}\left[\left(\int_0^1 B_t dt\right)^2\right] \\
=& \int_0^1 \int_0^1 R_B(s,t) ds dt = \frac{1}{12}
\end{aligned}$$

X and W_1 have covariance

$$\begin{aligned}
\mathrm{Cov}(X, W_1) =& \mathbb{E}\left[XW_1\right] \\
=& \mathbb{E}\left[W_1 \int_0^1 \left(W_t - tW_1\right) dt\right] \\
=& \mathbb{E}\left[\int_0^1 W_1 W_t dt - W_1 W_1 \int_0^1 t dt\right] \\
=& \int_0^1 \mathbb{E}\left[W_1 W_t\right] dt - \mathbb{E}\left[W_1 W_1\right] \int_0^1 t dt \\
=& \int_0^1 R_W(1,t) dt - \frac{1}{2} R_W(1,1) \\
=& \int_0^1 t dt - \frac{1}{2} \\
=& \frac{1}{2} - \frac{1}{2} = 0
\end{aligned}$$

X and W_1 are uncorrelated and therefore independent. They have a joint Gaussian distribution:

$$\begin{pmatrix} X \\ W_1 \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{matrix} \frac{1}{12} & 0 \\ 0 & 1 \end{matrix}\right)$$