ECE 534 RANDOM PROCESSES **FALL 2012**
SOLUTIONS FOR PROBLEM SET 5

1 (4.31) Mean hitting time for a discrete-time, discrete-state Markov process

(a) The transition probability matrix P can be written as:

$$P = \begin{pmatrix} .6 & .4 & 0 \\ .8 & 0 & .2 \\ 0 & .4 & .6 \end{pmatrix}$$

(b) In order to get the invariant distribution, we have to find a row matrix π such that $\pi P = \pi$. Therefore:

$$\begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix} \begin{pmatrix} .6 & .4 & 0 \\ .8 & 0 & .2 \\ 0 & .4 & .6 \end{pmatrix} = \begin{pmatrix} \pi_1 & \pi_2 & \pi_3 \end{pmatrix}$$

Solving the system of equations:

$$\begin{aligned} \pi_1 &= \frac{4}{7} \\ \pi_2 &= \frac{2}{7} \\ \pi_3 &= \frac{1}{7} \\ \pi &= \begin{pmatrix} \frac{4}{7}, & \frac{2}{7}, & \frac{1}{7} \end{pmatrix} \end{aligned}$$

(c) Suppose that we have $\tau = \min\{k \geq 0 : X_k = 3\}$, and define $a_i = \mathbb{E}[\tau \mid X_0 = i]$. Then we have:

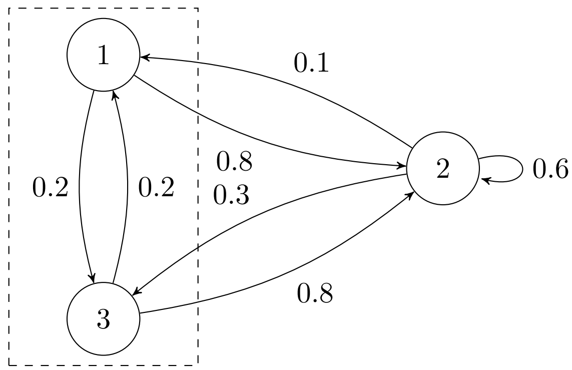
$$\begin{cases} a_1 = & .6(1 + a_1) + .4(1 + a_2) + 0(1 + a_3) = 1 + .6a_1 + .4a_2 \\ a_2 = & .8(1 + a_1) + .2(1 + a_3) = 1 + .8a_1 \end{cases}$$

Solving the system we have:

$$\begin{cases} a_1 = & 17.5 \\ a_2 = & 15 \end{cases}$$

2 (4.37) A state space reduction preserving the Markov property

(a) The transition probability diagram is as follows:



The invariant distribution can be computed as:

$$\pi \begin{pmatrix} 0.0 & 0.8 & 0.2 \\ 0.1 & 0.6 & 0.3 \\ 0.2 & 0.8 & 0.0 \end{pmatrix} = \pi$$

Solving the system we have:

$$\pi = \left(\tfrac{1}{9}, \quad \tfrac{2}{3}, \quad \tfrac{2}{9} \right)$$

(b) Notice that $P_{1,2} = P_{3,2} = 0.8$ and $P_{1,3} = P_{3,1} = 0.2$. If the process is in either of those two states, it has the same probability of leaving the pair of states and the same probability of staying in the pair of states. Therefore states 2 and 3 can be grouped together without changing the behavior of the process.

$$f(s) = \begin{cases} a, & s = 2 \\ b, & s \in \{1,3\} \end{cases}$$

$$P_Y = \begin{pmatrix} .6 & .4 \\ .8 & .2 \end{pmatrix}$$

3 (1) A compound process

(a) The expectation and covariance matrix of the process can be computed as follows, mainly using the tower property:

$$\begin{aligned}\mu_Y(t) =& \mathbb{E}\left[\sum_{i=1}^{N_t} X_i\right] \\ =& \mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^{N_t} X_i \mid N_t\right]\right] \\ =& \mathbb{E}\left[N_t \mathbb{E}\left[X_i\right]\right] = 0\end{aligned}$$

$$\begin{aligned}C_Y(s,t) =& \mathbb{E}\left[\left(Y_s - \mu_Y(s)\right)\left(Y_t - \mu_Y(t)\right)\right] \\ =& \mathbb{E}\left[Y_s Y_t\right] \\ =& \mathbb{E}\left[\left(\sum_{i=1}^{N_s} X_i\right)\left(\sum_{i=1}^{N_t} X_i\right)\right] \\ =& \mathbb{E}\left[\left(\sum_{i=1}^{N_s} X_i\right)\left(\sum_{i=1}^{N_s} X_i + \sum_{i=N_s+1}^{N_t} X_i\right)\right] \\ =& \mathbb{E}\left[\left(\sum_{i=1}^{N_s} X_i\right)^2\right] + \mathbb{E}\left[\left(\sum_{i=1}^{N_s} X_i\right)\left(\sum_{i=N_s+1}^{N_t} X_i\right)\right] \\ =& \mathbb{E}\left[\sum_{i=1}^{N_s}\sum_{j=1}^{N_s} X_i X_j\right] + \mathbb{E}\left[\sum_{i=1}^{N_s}\sum_{j=N_s+1}^{N_t} X_i X_j\right] \\ =& \mathbb{E}\left[\sum_{i=1}^{N_s} X_i^2\right] + 0 \\ =& \mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^{N_s} X_i^2 \mid N_s\right]\right] \\ =& \mathbb{E}\left[N_s\right] = \lambda s\end{aligned}$$

(b) Yes. It has independent increments. Basically, suppose that we have a sequence $t_1 < t_2 < \cdots < t_n < t$. Then

$$Y_t = \sum_{i=1}^{N_t} X_i = Y_{t_n} + D_t = Y_{t_n} + \begin{cases} 0, & \text{If } N_t = N_{t_n} \\ \cdots + X_{N_t}, & \text{If } N_t > N_{t_n} \end{cases} \tag{1}$$

From this we have:

$$Y_t - Y_{t_n} = D_t = \begin{cases} 0, & \text{If } N_t = N_{t_n} \\ \cdots + X_{N_t}, & \text{If } N_t > N_{t_n} \end{cases} \tag{2}$$

Which does not depend on the history of Y process, first because poisson process has independent increaments and second because of i.i.d. property of X_ns.

(c) Yes. The process is a martingale. To see why, we refer back to equation (**??**):

$$\begin{aligned}\mathbb{E}\left[Y_t \mid Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n}\right] =& \mathbb{E}\left[Y_{t_n} + D_t \mid Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n}\right] \\ =& Y_{t_n} + \mathbb{E}\left[D_t \mid Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n}\right]\end{aligned}$$

However, it can be easily shown that $\mathbb{E}[D_t] = 0$ by tower property.

4 (2) Random walk on the N-cycle

(a) It's very easy to see that the chain is equivalent to:

$$X_n = (X_{n-1} + U_n) \mod N$$

(b) The chain forms a ring, so there is a path from any state to any other state. To be more rigorous, note that for any $i < j \in \mathcal{S}$, we have $P_{ij} \geq P_{i(i+1)} P_{(i+1)(i+2)} \cdots P_{(j-1)(j)} = \frac{1}{2^{j-i}} > 0$.

(c) Because the chain forms a ring, there are two ways for the process to return to state i: by leaving to another state and returning back through the same state, and by traversing the entire ring and returning through the other adjacent state. In the first case, excursion always takes an even number of jumps ($r = m$ jumps moving clockwise plus m jumps moving counterclockwise which adds up to $2m$ total jumps). In the second, the excursion takes $r = N + 2m$ jumps, where m is the number of backward-forward jumps. The period is then

$$\begin{aligned}\text{GCD}\left\{r \geq 0 : p_{ii}^r > 0\right\} &= \text{GCD}\left(\{r = 2m,\ m = 0, 1, 2, \ldots\} \cup \{r = N + 2m,\ m = 0, 1, 2, \ldots\}\right) \\ &= \begin{cases} 1, & N \text{ odd} \\ 2, & N \text{ even} \end{cases}\end{aligned}$$

so X is periodic with period 2 if N is even and aperiodic if N is odd.

(d) For fixed i and j, $p_{ij}^r > 0$ if r is of the form

$$\begin{aligned} r &= |i - j| + 2m, & m = 0, 1, 2, \ldots \\ \text{or } r &= N - |i - j| + 2m, & m = 0, 1, 2, \ldots \end{aligned}$$

Since N is odd, one of the two above expressions covers all sufficiently large even values of r and the other covers all sufficiently large odd values of r. Therefore $p_{ij}^r > 0$ for all $r \geq$

$\max(|i-j|, N-|i-j|) - 1$. Considering all pairs (i, j), a lower bound on r is:

$$\begin{aligned} r &\geq \max_{i,j \in S} [\max(|i-j|, N-|i-j|) - 1] \\ &= N-1 \end{aligned}$$

If r is at least $N-1$, there will always be a path from one state to any other state. It must now be shown that for r less than $N-1$, there is always some (i, j) such that $p_{ij}^r = 0$. If $r < N-1$ and r is even, then there is no path from $i = 0$ to $i = 1$ in r steps. If $r < N-1$ and r is odd, then there is no path from $i = 0$ to $j = 0$ in r steps. Therefore, the smallest value of r such that $p_{ij}^r > 0$ for all $i, j \in S$ is $r = N-1$.

5 (3) Random walk on a simple undirected gtaph

(a) First, we check to see that the vector π with $\pi_i = \deg(i)/2|E|$ is a valid probability distribution. We have $\sum_{i\in V} \frac{\deg(i)}{2|E|} = \frac{\sum_{i\in V}\deg(i)}{2|E|} = \frac{2|E|}{2|E|} = 1$. According to definition, it is a stationary distribution only if it is invariant, $\pi P = \pi$. For each state $i \in V$,

$$\begin{aligned} (\pi P)_i &= \sum_{j\in V} \pi_j p_{ji} \\ &= \sum_{j\in V} \pi_j \frac{1}{\deg(j)} I_{\{(i,j)\in E\}} \\ &= \sum_{j\in V} \frac{\deg(j)}{2|E|} \frac{1}{\deg(j)} I_{\{(i,j)\in E\}} \\ &= \frac{1}{2|E|} \sum_{j\in V} I_{\{(i,j)\in E\}} \\ &= \frac{\deg(i)}{2|E|} = \pi_i \end{aligned}$$

Therefore $\pi P = \pi$, so π is a stationary distribution.

(b) Each edge $(i_l, i_{l+1}) \in E$ represents a transition of the Markov process X with state space $S = V$ that has positive probability. The sequence of vertices $i_0, i_1, \ldots, i_k$ corresponds to a sequence of states leading from i to j. Thus if there is a sequence of edges from i to j for all $i, j \in V$, then there is a sequence of state transitions leading from state i to state j with $p_{ij}^k > 0$ for all $i, j \in S$. If there is no such sequence of vertices, then $p_{ij}^k = 0$. Thus, X is irreducibleif and only if G is connected.