**ECE 534 RANDOM PROCESSES** **FALL 2012**
**SOLUTIONS FOR PROBLEM SET 4**

## 1 (4.3) A sinusoidal random process

For the mean function we write:

$$\begin{aligned}\mu_X(t) &= \mathbb{E}\left[A\cos(2\pi Vt+\Theta)\right]\\ &= \mathbb{E}[A]\mathbb{E}\left[\int_{[0,2\pi]}\cos(2\pi Vt+\theta)f_\Theta(\theta)d\theta\right]\\ &= 2\ \mathbb{E}\left[\int_{[0,2\pi]}\cos(2\pi Vt+\theta)\frac{d\theta}{2\pi}\right]\\ &= 0\end{aligned}$$

For the Autocorrelation function:

$$\begin{aligned}R_X(s,t) &= \mathbb{E}[X_sX_t]\\ &= \mathbb{E}\left[A^2\ \cos(2\pi Vt+\Theta)\ \cos(2\pi Vs+\Theta)\right]\\ &= \mathbb{E}[A^2]\ \frac{1}{2}\ \mathbb{E}\left[\cos(2\pi V(s-t))+\cos(2\pi V(s+t)+2\Theta)\right]\\ &= 4\ \mathbb{E}\left[\cos(2\pi V(s-t))\right]+4\mathbb{E}\left[\int_{[0,2\pi]}\cos(2\pi Vt+\theta)\frac{d\theta}{2\pi}\right]\\ &= 4\ \mathbb{E}\left[\cos(2\pi V(s-t))\right]\\ &= 4\ \int_0^5\cos(2\pi v(s-t))\frac{dv}{5}\\ &= 4\frac{\sin(10\pi(s-t))}{10\pi(s-t)}\end{aligned}$$

## 2 (4.11) A simple Discrete-time random process

(a) See figure 1.

(b) The process is stationary. The reason is that $X_k$ is a time-invariant deterministic fucntion of $U_k$ which is an i.i.d. process. More formally:

$$\begin{aligned}&F_{X_{k_1},X_{k_2},\ldots,X_{k_n}}(a_1,a_2,\ldots,a_n)\\ =&\Pr(\max\{U_{k_1-1},U_{k_1}\}\le a_1,\max\{U_{k_2-1},U_{k_2}\}\le a_2,\ldots,\max\{U_{k_1-1},U_{k_1}\}\le a_n)\\ =&\Pr(\max\{U_{k_1+p-1},U_{k_1+p}\}\le a_1,\max\{U_{k_2+p-1},U_{k_2+p}\}\le a_2,\ldots,\max\{U_{k_1+p-1},U_{k_1+p}\}\le a_n)\\ =&F_{X_{k_1+p},X_{k_2+p},\ldots,X_{k_n+p}}(a_{k_1},a_2,\ldots,a_n)\end{aligned}$$

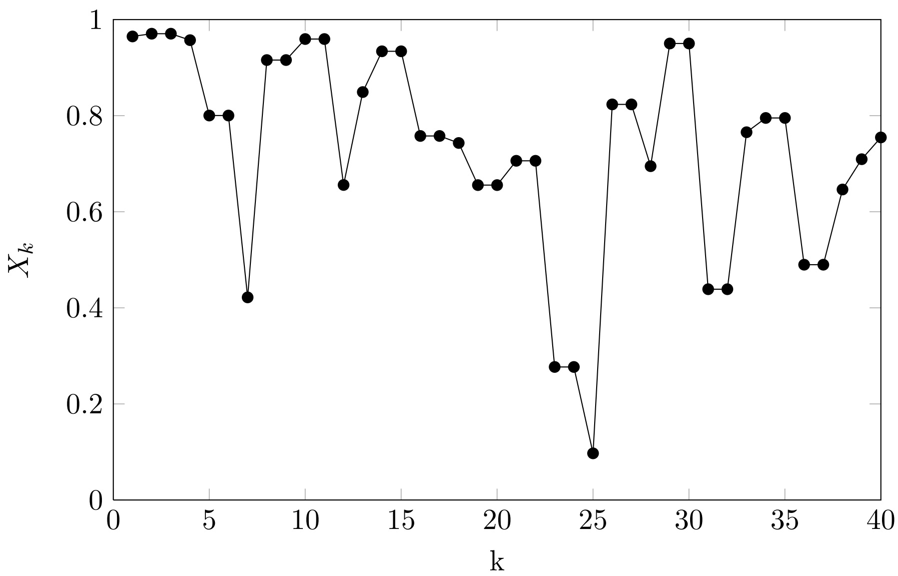


Figure 1: A sample path of $X$ for $k = 1, \ldots, 30$. The values often come in pairs, and are more dense near 1 than 0.

(c) The process is not Markov. To see why, consider the below example:

$$\Pr(\max\{U_k, U_{k-1}\} > c \mid \max\{U_{k-1}, U_{k-2}\} > c \ , \ \max\{U_{k-2}, U_{k-3}\} < c) = 1$$
$$\Pr(\max\{U_k, U_{k-1}\} > c \mid \max\{U_{k-1}, U_{k-2}\} > c \ , \ \max\{U_{k-2}, U_{k-3}\} > c) \neq 1$$

(d) First Order Distribution

$$F_{X_n}(c) = \Pr(X_n \leq c) = \Pr(\max\{U_{n-1}, U_n\} \leq c) = \begin{cases} 0, & c < 0 \\ c^2, & 0 \leq c < 1 \\ 1, & c \geq 1 \end{cases}$$

(e) Second Order Distribution

$$
\begin{aligned}
&F_{X_n,X_m}(c_1,c_2) = \Pr(X_n \le c_1, X_m \le c_2)\\
=&\Pr(\max\{U_{n-1},U_n\} \le c_1, \max\{U_{m-1},U_m\} \le c_2)\\
=&\Pr(U_{n-1} \le c_1, U_n \le c_1, U_{m-1} \le c_2, U_m \le c_2)\\
=&\begin{cases}
0, & c_1 < 0 \text{ or } c_2 < 0\\
c_1^2c_2^2, & |n-m| > 1 \text{ and } 0 \le c_1 < 1 \text{ and } 0 \le c_2 < 1\\
c_1c_2\min\{c_1,c_2\}, & |n-m| = 1 \text{ and } 0 \le c_1 < 1 \text{ and } 0 \le c_2 < 1\\
c_1^2, & 0 \le c_1 < 1 \text{ and } 1 \le c_2\\
c_2^2, & 1 \le c_1 \text{ and } 0 \le c_2 < 1\\
1, & 1 \le c_1 \text{ and } 1 \le c_2
\end{cases}
\end{aligned}
$$

## 3 (4.13) Sliding function of an i.i.d. Poisson Sequence

(a) For each $k$, $X_k$ and $X_{k+1}$ are i.i.d poisson processes and we have

$$
\begin{aligned}
&f_{Y_k}(n) = \Pr(Y_k = n) = \Pr(X_k + X_{k+1} = n)\\
=&\sum_{m=0}^{n} f_{X_k}(n) f_{X_{k+1}}(n-m) = \sum_{m=0}^{n} \exp(-\lambda)\frac{(\lambda)^m}{m!}\exp(-\lambda)\frac{(\lambda)^{n-m}}{(n-m)!}\\
=&\frac{\exp(-2\lambda)\lambda^n}{n!}\sum_{m=0}^{n}\frac{n!}{m!(n-m)!}\\
=&\frac{\exp(-2\lambda)\lambda^n}{n!}\sum_{m=0}^{n}\binom{n}{m}\\
=&\frac{\exp(-2\lambda)\lambda^n}{n!}2^n = \frac{\exp(-2\lambda)(2\lambda)^n}{n!}
\end{aligned}
$$

Which means that $Y_k$ is a poisson process with parameter $2\lambda$.

(b) $\{X\}_k$ is an i.i.d. process and therefore is a stationary one. As a general proof:

$$
\begin{aligned}
F_{X_{k_1},X_{k_2},\cdots,X_{k_n}}(c_1,c_2,\cdots,c_n) =&F_{X_{k_1}}(c_1)\ F_{X_{k_2}}(c_2)\ \cdots\ F_{X_{k_n}}(c_n)\\
=&F_{X_{k_1+p}}(c_1)\ F_{X_{k_2+p}}(c_2)\ \cdots\ F_{X_{k_n+p}}(c_n)\\
=&F_{X_{k_1+p},X_{k_2+p},\cdots,X_{k_n+p}}(c_1,c_2,\cdots,c_n)
\end{aligned}
$$

(c) $Y_k$ is also a stationary process. The reason is that $Y_k$ is a time-invariant deterministic fucntion of $X_k$ which is an i.i.d. process and the proof is like what we had in part (b) of question (4.11).

## 4 (4.25) Identification of special properties of two discrete-time processes

(a)

1. The process is Markov. The reason is that $X_{k+1} = (1 + X_k)U_k$ is independent of $X_{k-1}$,...,$X_1$, given $X_k$.
2. The process is not a martingal:

$$\begin{aligned}\mathbb{E}\left[X_k|X_{k-1} = x_{k-1}\right] =& \mathbb{E}\left[(1 + X_{k-1})U_{k-1}|X_{k-1} = x_{k-1}\right]\\ =& \mathbb{E}\left[(1 + x_{k-1})U_{k-1}\right]\\ =& \frac{1}{2}(1 + x_{k-1}) \neq x_{k-1}\end{aligned}$$

3. The proecss does not have independent increment.

$$X_{k+1} - X_k = (1 + X_k)U_k - X_k = U_k + (U_k - 1)X_k$$

We can see that the increament depends on the value of $X_k$.

(b)

1. The process is not Markov. One way to show this is to consider the expectation of $Y_k$:

$$\begin{aligned}\mathbb{E}\left[Y_{k+1}|Y_0, \cdots, Y_k\right] =& \mathbb{E}\left[V_{k+1} + V_k + V_{k-1}|Y_0, \cdots, Y_k\right]\\ =& 0 + \mathbb{E}\left[V_k + V_{k-1}|Y_0, \cdots, Y_k\right]\\ =& Y_k - \mathbb{E}\left[V_{k-2}|Y_0, \cdots, Y_k\right] = Y_k - V_{k-2}\end{aligned} \quad (1)$$

The reason is that $V_0, \cdots, V_k$ can be calculated given $Y_0, \cdots, Y_k$. This shows that given $Y_k$, the expectation of $Y_{k+1}$ depends on past values of $Y$ sequence.

2. The process is not a martingale. The reason can be seen in (1).
3. The proecss does not have independent increments. Basically, as an example, the increment $Y_2 - Y_1 = V_2$. The increment $Y_5 - Y_4 = (V_5 + V_4 + V_3) - (V_4 + V_3 + V_2) = V_5 - V_2$, which shows that the second increment depends on the first, so they are not independent.

## 5 (4.27) Identification of special properties of two continuous-time processes

(a)

1. The process is Markov, since for $\forall 0 < t_1 < t_2 < \cdots < t_n < t$

$$\begin{aligned}
&\Pr(Z_t = c | Z_{t_n} = c_n, Z_{t_{n-1}} = c_{n-1}, ..., Z_{t_1} = c_1) \\
=&\Pr(\exp(W_t - \frac{\sigma^2 t}{2}) = c | \exp(W_{t_n} - \frac{\sigma^2 t_n}{2}) = c_n, ... \exp(W_{t_1} - \frac{\sigma^2 t_1}{2}) = c_1)
\end{aligned}$$

Now, notice that:

$$\begin{aligned}
Z_t =& \exp(W_t - \frac{\sigma^2 t}{2}) \\
=& \exp(W_{t_n} + W_t - W_{t_n} - \left(\frac{\sigma^2 t_n}{2} + \frac{\sigma^2 t}{2} - \frac{\sigma^2 t_n}{2}\right)) \\
=& \exp(W_{t_n} - \frac{\sigma^2 t_n}{2}) \exp((W_t - W_{t_n}) - \frac{\sigma^2 (t - t_n)}{2})
\end{aligned} \tag{2}$$

This concludes that

$$\begin{aligned}
&\Pr(\exp(W_t - \frac{\sigma^2 t}{2}) = c | \exp(W_{t_n} - \frac{\sigma^2 t_n}{2}) = c_n, ... \exp(W_{t_1} - \frac{\sigma^2 t_1}{2}) = c_1) \\
=&\Pr(\exp(W_{t_n} - \frac{\sigma^2 t_n}{2}) \exp((W_t - W_{t_n}) - \frac{\sigma^2 (t - t_n)}{2}) = c | \exp(W_{t_n} - \frac{\sigma^2 t_n}{2}) = c_n, ... \exp(W_{t_1} - \frac{\sigma^2 t_1}{2}) = c_1) \\
=&\Pr(\exp((W_t - W_{t_n}) - \frac{\sigma^2 (t - t_n)}{2}) = \frac{c}{c_n} | \exp(W_{t_n} - \frac{\sigma^2 t_n}{2}) = c_n, ... \exp(W_{t_1} - \frac{\sigma^2 t_1}{2}) = c_1) \\
=&\Pr(\exp((W_t - W_{t_n}) - \frac{\sigma^2 (t - t_n)}{2}) = \frac{c}{c_n})
\end{aligned}$$

2. The process is a martingal. According to (2):

$$\begin{aligned}
\mathbb{E}\left[Z_t | Z_{t_1}, \cdots, Z_{t_n}\right] =& \mathbb{E}\left[Z_{t_n} e^{(W_t - W_{t_n}) - \frac{\sigma^2 (t - t_n)}{2}} | Z_{t_1}, \cdots, Z_{t_n}\right] \\
=& Z_{t_n} \mathbb{E}\left[e^{(W_t - W_{t_n}) - \frac{\sigma^2 (t - t_n)}{2}}\right] \\
=& Z_{t_n} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 (t - t_n)}} e^{-\frac{x^2}{2\sigma^2 (t - t_n)}} e^{x - \frac{\sigma^2 (t - t_n)}{2}} dx \\
=& Z_{t_n} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 (t - t_n)}} e^{-\frac{x^2 - 2\sigma^2 (t - t_n) x + \left(\sigma^2 (t\text{-}t_n)\right)^2}{2\sigma^2 (t - t_n)}} dx \\
=& Z_{t_n} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2 (t - t_n)}} e^{-\frac{\left(x - \sigma^2 (t - t_n)\right)^2}{2\sigma^2 (t - t_n)}} dx \\
=& Z_{t_n}
\end{aligned}$$

3. The proecss does not have independent increment.

$$
\begin{aligned}
Z_{t_{n+1}} - Z_{t_n} =& Z_{t_n} e^{\left(W_{t_{n+1}} - W_{t_n}\right) - \frac{\sigma^2 (t_{n+1} - t_n)}{2}} - Z_{t_n} \\
=& Z_{t_n} \left( e^{\left(W_{t_{n+1}} - W_{t_n}\right) - \frac{\sigma^2 (t_{n+1} - t_n)}{2}} - 1 \right)
\end{aligned}
$$

which depends on $Z_{t_n}$, which itself depends on past increments. Therefore the increments are not independent.

(b) $R_t$ is the sum of iid random variables that arrive according to a Poisson process with rate $\lambda$; say: $R_t = D_1 + D_2 + \cdots + D_{N_t}$.

1. This is a Markov process. In order to see why, notice that if we have a sequence $t_1 < t_2 < \cdots < t_n < t$ then:

$$
R_t = D_1 + D_2 + \cdots + D_{N_t} = \begin{cases} R_{t_n}, & \text{if } N_t = N_{t_n} \\ R_{t_n} + (\cdots + D_{N_t}), & \text{if } N_t > N_{t_n} \end{cases} \tag{3}
$$

Which shows that $R_t$ depends on all history of $R$ only through $R_{t_n}$. Notice that if $N_t > N_{t_n}$, both $N_{t_n} - N_t$ and $D$'s are independent of $R_{t_n}$ and past values of $R$.

2. Referring back to (3) we see that if we have a sequence $t_1 < t_2 < \cdots < t_n < t$ then:

$$
\begin{aligned}
\mathbb{E}\left[R_t | R_{t_1} \cdots, R_{t_n}\right] =& \mathbb{E}\left[R_{t_n} + \cdots + D_{N_t} | R_{t_1} \ldots, R_{t_n}\right] \\
=& R_{t_n} + \mathbb{E}\left[\cdots + D_{N_t} | R_{t_1} \ldots, R_{t_n}\right] = R_{t_n}
\end{aligned}
$$

so the martingale property does hold. The reason is that $D$ is a mean zero random variable, and the second expectation can be easily proved to be zero according to tower property.

3. $R_t$ is an independent increament random process. This can be proved using (3) as we have:

$$
R_t - R_{t_n} = \begin{cases} 0, & \text{if } N_t = N_{t_n} \\ \cdots + D_{N_t}, & \text{if } N_t > N_{t_n} \end{cases}
$$

and $D$'s are independent of past values of $R_t$.

## 6 (1) Two random processes

(a)

For the first random process we have $X_t(\omega) = \begin{cases} 1, & \text{if } 0 < t \leq \omega \\ 0, & \text{otherwise} \end{cases}$

Therefore:

$$\begin{aligned} p_{X_t}(1) =& P\{\omega : \omega \geq t\} = \begin{cases} 1-t, & \text{if } 0 < t \leq 1 \\ 0, & \text{otherwise} \end{cases} \\ p_{X_t}(0) =& P\{\omega : \omega < t\} = \begin{cases} t, & \text{if } 0 < t \leq 1 \\ 1, & \text{otherwise} \end{cases} \end{aligned}$$

(b)

For the second process we have $Y_t(\omega) = \begin{cases} t/\omega, & \text{if } 0 < t \leq \omega \\ 0, & \text{otherwise} \end{cases}$

Therefore:

$$\begin{aligned} P(Y_t < y) = P\left(0 < t \leq \omega, \frac{t}{\omega} \leq y\right) =& P\left(\omega \geq t, \omega \geq \frac{t}{y}\right) \\ =& P\left\{\omega : \max\left\{\frac{t}{y}, t\right\} \leq \omega\right\} \\ =& \begin{cases} 1 - \frac{t}{y}, & \text{if } t < y \leq 1 \\ 1 - t, & \text{if } t < 1 < y \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

## 7 (2) Standardized random walk

We have $Y_n = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} X_k$. For the characteristic function:

$$
\begin{aligned}
\Phi_{Y_n}(u) &= \mathbb{E}\left[e^{juY_n}\right] \\
&= \mathbb{E}\left[e^{ju\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} X_k}\right] \\
&= \mathbb{E}\left[\prod_{k=0}^{n-1} e^{ju\frac{1}{\sqrt{n}}X_k}\right] \\
&= \prod_{k=0}^{n-1} \mathbb{E}\left[e^{ju\frac{1}{\sqrt{n}}X_k}\right] \\
&= \left(\mathbb{E}\left[e^{ju\frac{1}{\sqrt{n}}X}\right]\right)^n \\
&= \left(p_X(-1)e^{ju\frac{1}{\sqrt{n}}(-1)} + p_X(1)e^{ju\frac{1}{\sqrt{n}}(1)}\right)^n \\
&= \left(\frac{1}{2}e^{-ju\frac{1}{\sqrt{n}}} + \frac{1}{2}e^{ju\frac{1}{\sqrt{n}}}\right)^n \\
&= \left(\cos\frac{u}{\sqrt{n}}\right)^n \\
&= e^{\log\left(\cos\frac{u}{\sqrt{n}}\right)^n} \\
&= e^{n\log\left(\cos\frac{u}{\sqrt{n}}\right)}
\end{aligned}
$$

The limit of the characteristic function is found by taking a second order Taylor series of the cosine function near zero:

$$
\begin{aligned}
\lim_{n\to\infty} \Phi_{Y_n}(u) &= \lim_{n\to\infty} \left(\cos\frac{u}{\sqrt{n}}\right)^n \\
&= \lim_{n\to\infty} \left(1 - \frac{1}{2}\left(\frac{u}{\sqrt{n}}\right)^2\right)^n \\
&= \lim_{n\to\infty} \left(1 + \frac{-u^2/2}{n}\right)^n \\
&= e^{-u^2/2}
\end{aligned}
$$

Which is the characteristic function of the standard normal. Note that this result confirms centeral limit theorem.

## 8 (3) Multiplication by random signs

**(a)** The autocorrelation function is

$$R_Y(k,l) = \mathbb{E}\left[Y_k Y_l\right] = \mathbb{E}\left[U_k X_k U_l X_l\right] = \mathbb{E}\left[U_k U_l\right]\mathbb{E}\left[X_k X_l\right] = \begin{cases} 0, & k \neq l \\ \sigma^2, & k = l \end{cases}$$

**(b)** The characteristic function is

$$\begin{aligned}
\Phi_{Y_n}(u) &= \mathbb{E}\left[e^{juY_n}\right] \\
&= \mathbb{E}\left[e^{juY_n}|U_n = 1\right] P(U_n = 1) + \mathbb{E}\left[e^{juY_n}|U_n = -1\right] P(U_n = -1) \\
&= \frac{1}{2}\mathbb{E}\left[e^{juX_n}\right] + \frac{1}{2}\mathbb{E}\left[e^{-juX_n}\right] \\
&= \frac{1}{2}e^{-\frac{1}{2}\sigma^2 u^2} + \frac{1}{2}e^{-\frac{1}{2}\sigma^2(-u)^2} \\
&= e^{-\frac{1}{2}\sigma^2 u^2}
\end{aligned}$$

This shows that $Y_n$ has a characteristic function of a standard Gaussian.

**(c)** Yes. Since each $Y_n$ is multiplication of two i.i.d random random variables, $Y_n$'s have the same distribution and are independant.

**(d)** As we see from part (b), $Y_n$ is a Gaussian random variable with distribution $\mathcal{N}(0, \sigma^2)$. So by the law of large numbers, the sample averages converge in the mean square sense to the mean, which is zero.

$$\bar{Y}_n \overset{\text{m.s.}}{\rightarrow} 0$$