1 Stochastic convergence of sample variance

(a) We want to prove that $\mathbb{E}[\mathbb{V}_n] = \sigma^2$.

$$
\mathbb{E}[\mathbb{V}_n] = \frac{1}{n-1} \mathbb{E} \left[ \sum_{i=1}^{n} (X_i - \overline{X}_n)^2 \right]
$$

$$
= \frac{1}{n-1} \mathbb{E} \left[ \sum_{i=1}^{n} (X_i - \mu + \mu - \overline{X}_n)^2 \right]
$$

$$
= \frac{1}{n-1} \left[ n\sigma^2 \right] + \frac{1}{n-1} \sum_{i=1}^{n} \mathbb{E} \left[ \left( \frac{\sum_{j=1}^{n} X_j - n\mu}{n} \right)^2 \right] - \frac{2}{n-1} \sum_{i=1}^{n} \mathbb{E} \left[ (X_i - \mu)(\sum_{j=1}^{n} X_j - n\mu) \right]
$$

$$
= \frac{1}{n-1} \left[ n\sigma^2 \right] + \frac{1}{(n-1)n^2} \sum_{i=1}^{n} \mathbb{E} \left[ \left( \sum_{j=1}^{n} (X_j - \mu) \right)^2 \right] - \frac{2}{(n-1)n} \sum_{i=1}^{n} \mathbb{E} \left[ (X_i - \mu) \left( \frac{\sum_{j=1}^{n} X_j - n\mu}{n} \right) \right]
$$

$$
= \frac{1}{n-1} \left[ n\sigma^2 \right] + \frac{1}{(n-1)n^2} \sum_{i=1}^{n} \mathbb{E} \left[ \sum_{j=1}^{n} (X_j - \mu)^2 \right] - \frac{2}{(n-1)n} \sum_{i=1}^{n} \mathbb{E} \left[ (X_i - \mu) \left( \frac{\sum_{j=1}^{n} X_j - n\mu}{n} \right) \right]
$$

$$
= \frac{1}{n-1} \left[ n\sigma^2 \right] + \frac{1}{(n-1)n^2} \left[ n^2 \sigma^2 \right] - \frac{2}{(n-1)n} \left[ n\sigma^2 \right]
$$

$$
= \frac{n}{n-1} \left[ \sigma^2 \right] + \frac{1}{(n-1)} \left[ \sigma^2 \right] - \frac{2}{(n-1)} \left[ \sigma^2 \right] = \sigma^2
$$

(b) In order to show how it converges to the same value in probability sense:
Hereafter, it is straightforward. According to S.L.L.N., we know that
\[ \sum_{i=1}^{n} \frac{X_i^2}{n} \xrightarrow{a.s.} \sigma^2 + \mu^2 \] and
\[ \frac{\sum_{i=1}^{n} X_i}{n} \xrightarrow{a.s.} \mu \] which conclude convergence in probability.

By writing the definition of convergence in probability, we can see that if \( a_n \xrightarrow{p} a \) and \( b_n \xrightarrow{p} b \) then first, \( a_n^2 \xrightarrow{p} a^2 \) and second, \( a_n + b_n \xrightarrow{p} a + b \). Moreover, since \( \frac{n}{n-1} \to 1 \), we have \( \nabla_n \xrightarrow{p} \sigma^2 + \mu^2 - \mu^2 = \sigma^2 \).

### 2 Wild oscillations

First, we review the convergence in probability sense. For every \( \epsilon > 0 \) we have \( \mathbb{P}\{|X_n - 0| > \epsilon\} = \frac{1}{n^2} \).
It can be easily concluded that \( \lim_{n \to \infty} \mathbb{P}\{|X_n - 0| > \epsilon\} = 0 \). Therefore, \( X_n \xrightarrow{p} 0 \).

Convergence in distribution can also be concluded from the convergence in probability.

For considering convergence in *almost sure* sense, note that the probability space \( \{\Omega, \mathcal{F}, \mathbb{P}\} \) is not given to us. Therefore, we cannot directly apply the definition of almost sure convergence. However, we can use Borell-Cantelli lemma to evaluate it:

For each \( 1 > \epsilon > 0 \) Define:
\[
A_n^\epsilon = \{\omega \in \Omega \text{ s.t. } |X_n(\omega)| > \epsilon\}
\]

We want to prove that
\[ \mathbb{P}(\cap_{m=0}^{\infty} \cup_{n=m}^{\infty} A_n^\epsilon) = 0 \]

Which is equivalent to the definition of the convergence in almost sure sense. According to Borell-Cantelli lemma, we only need to consider that:
\[
\sum_{n=0}^{\infty} \mathbb{P}(A_n^\epsilon) = \sum_{n=0}^{\lfloor \frac{1}{\epsilon} \rfloor} 1 + \sum_{n=\lfloor \frac{1}{\epsilon} \rfloor}^{\infty} \frac{1}{n^2} < \infty
\]
For convergence in mean square sense, we re-write the definition:

\[
\mathbb{E}[(X_n - 0)^2] = \mathbb{E}[X_n^2] = \frac{1}{n^2} \mathbb{P}\{X_n = \frac{1}{n}\} + n^2 \mathbb{P}\{X_n = n\}
\]

\[
= \frac{1}{n^2}(1 - \frac{1}{n^2}) + n^2 \frac{1}{n^2}
\]

\[
= 1 + \frac{1}{n^2} - \frac{1}{n^4}
\]

This means that

\[
\lim_{n \to \infty} \mathbb{E}[(X_n - 0)^2] = 1 \neq 0
\]

And we do not have convergence in mean-square sense.

### 3 Convergence to a constant in mean square sense

We first prove the ”if” part. It’s just enough to remember the definition of convergence in mean-square sense. We say \(\{X_n\}\) converges to \(X\) in mean square sense if \(\mathbb{E}[X_n^2] < \infty\) for all \(n\) and

\[
\lim_{n \to \infty} \mathbb{E}[(X_n - X)^2] = 0.
\]

Since \(\lim_{n \to \infty} \text{Var}[X_n]\) exists, we can conclude that it is bounded for all \(n\). Therefore, \(\mathbb{E}[X_n^2] = \text{Var}[X_n] + \mathbb{E}[X_n]^2\) is bounded.

We now take \(X = c\) in the definition of mean square convergence:

\[
\lim_{n \to \infty} \mathbb{E}[(X_n - c)^2] = \lim_{n \to \infty} \mathbb{E}[(X_n - \mathbb{E}[X_n])^2] = \lim_{n \to \infty} \text{Var}[X_n] = 0
\]

For the ”only if” part;

\[
\lim_{n \to \infty} \mathbb{E}[(X_n - c)^2] = \lim_{n \to \infty} \left[\mathbb{E}[X_n^2] + c^2 - 2c\mathbb{E}[X_n]\right]
\]

\[
= \lim_{n \to \infty} \left[\mathbb{E}[X_n^2] - \mathbb{E}[X_n]^2 + \mathbb{E}[X_n]^2 + c^2 - 2c\mathbb{E}[X_n]\right]
\]

\[
= \lim_{n \to \infty} \left[\text{Var}[X_n] + (\mathbb{E}[X_n] - c)^2\right]
\]

\[
= \lim_{n \to \infty} \text{Var}[X_n] + \lim (\mathbb{E}[X_n] - c)^2 = 0
\]

\[
\Rightarrow \begin{cases}
\lim_{n \to \infty} \text{Var}[X_n] = 0 \\
\lim_{n \to \infty} (\mathbb{E}[X_n] - c)^2 = 0
\end{cases}
\]

Where the last conclusion can be made because the only way for the sum of two non-negative terms to be zero is for both of them to be zero.
4 Stochastic convergence of a minimum

For this problem, we first need to find the probability distribution of \( X_n = n \min\{Z_1, \cdots, Z_n\} \).

\[
F_{X_n}(c) = P\{X_n \leq c\} = P\{\min\{Z_1, \cdots, Z_n\} \leq \frac{c}{n}\} \\
= 1 - P\{\min\{Z_1, \cdots, Z_n\} > \frac{c}{n}\} \\
= 1 - P\{Z_1 > \frac{c}{n}\}P\{Z_2 > \frac{c}{n}\} \cdots P\{Z_n > \frac{c}{n}\} \\
= 1 - \left(1 - F_Z\left(\frac{c}{n}\right)\right)^n \\
= 1 - \left(1 - \int_0^{\frac{c}{n}} f_Z(y)dy\right)^n
\]

Now, all we need to do is to prove that:

\[
\lim_{n \to \infty} 1 - \left(1 - \int_0^{\frac{c}{n}} f_Z(y)dy\right)^n = 1 - \exp(-\lambda c)
\]

We know that according to mean value theorem, there exist \( x_n \in (0, \frac{c}{n}) \) such that \( \int_0^{\frac{c}{n}} f_Z(y)dy = f_Z(x_n)\frac{c}{n} \). Therefore, we have:

\[
\lim_{n \to \infty} \left(1 - \int_0^{\frac{c}{n}} f_Z(y)dy\right)^n = \lim_{n \to \infty} \left(1 - c f_Z(x_n)\right)^n = \exp(-c \lim_{n \to \infty} f_Z(x_n)) = \exp(-\lambda c)
\]

Where the last conclusion can be made because as \( n \) goes to infinity, we have \( \frac{c}{n} \) thus \( x_n \) converging to zero form the right. We also know from the assumption that \( \lim_{x \to 0^+} f(x) = \lambda \) so \( \lim_{n \to \infty} f(x_n) = \lambda \).

5 Subgaussian random variables

(a) (i) \( X \) is subgaussian when \( X \sim \mathcal{N}(\mu, \sigma^2) \):

\[
\mathbb{E}[e^{t(X-EX)}] = \mathbb{E}[e^{tX} e^{-tEX}] \\
= \mathbb{E}[e^{tX}]e^{-t\mu} \\
= \Phi_X(-it)e^{-t\mu} \\
= e^{t\mu + \frac{t^2\sigma^2}{2}}e^{-t\mu} = e^{t\mu + \frac{t^2\sigma^2}{2}} \leq e^{\frac{t^2\sigma^2}{2}} \iff \sigma^2 \leq c
\]

(a) (ii) It is subgaussian. Note that \( EX = 0 \) thus \( \mathbb{E}[e^{t(X-EX)}] = \frac{e^{tM} + e^{-tM}}{2} \). Considering fourier expansion for \( e^{tM} \) we have:
\[ e^{tM} = 1 + \frac{tM}{1!} + \frac{(tM)^2}{2!} + \frac{(tM)^3}{3!} + \cdots \]
\[ e^{-tM} = 1 - \frac{tM}{1!} + \frac{(tM)^2}{2!} - \frac{(tM)^3}{3!} + \cdots \]
\[ \Rightarrow \frac{e^{tM} + e^{-tM}}{2} = 1 + \frac{(tM)^2}{2!} + \frac{(tM)^4}{4!} + \cdots \]
\[ \leq 1 + \frac{(tM)^2}{2!} + \frac{(tM)^4}{4!} + \cdots = e^{t^2M^2} \leq e^{ct^2} \iff M^2 \leq \frac{c}{2} \]

(a) (iii) It is not subgaussian. Note that \( \mathbb{E}X = \lambda \) when we have exponential distribution.

\[
\mathbb{E}[e^{t(X-\mathbb{E}X)}] = \mathbb{E}[e^{tX}e^{-t\mathbb{E}X}]
= \mathbb{E}[e^{tX}]e^{-t\lambda}
= \left[ \int_0^\infty e^{tx} \lambda e^{-\lambda x} dx \right] e^{-t\lambda}
= e^{-t\lambda} \int_0^\infty \lambda e^{(t-\lambda)x} dx
\]

The above integral diverges for \( t > \lambda \) and cannot be bounded by any function such as \( e^{ct^2} \) for all values of \( t \). Therefore, this r.v. is not subgaussian.

(b) First of all, note that \( \mathbb{E}Y = a_1\mathbb{E}X_1 + \cdots + a_n\mathbb{E}X_n \). We therefore have:

\[
\mathbb{E}\left[ e^{t(Y - \mathbb{E}Y)} \right] = \mathbb{E}\left[ e^{tY} \right] e^{-t\mathbb{E}Y}
= \mathbb{E}\left[ e^{t(a_1X_1 + \cdots + a_nX_n)} \right] e^{-t\mathbb{E}Y}
= \mathbb{E}\left[ e^{t(a_1X_1)} \right] \cdots \mathbb{E}\left[ e^{t(a_nX_n)} \right] e^{-ta_1\mathbb{E}X_1} \cdots e^{-ta_n\mathbb{E}X_n}
= \mathbb{E}\left[ e^{ta_1(X_1 - \mathbb{E}X_1)} \right] \cdots \mathbb{E}\left[ e^{ta_n(X_n - \mathbb{E}X_n)} \right]
\leq e^{\frac{(a_1t)^2}{2}} \cdots e^{\frac{(a_n t)^2}{2}}
= e^{\frac{t^2(a_1^2 + \cdots + a_n^2)}{2}} = e^{\frac{ct^2}{2}}
\]

(c) Again, like question 2, we use Borell-Cantelli Lemma for the proof of almost sure convergence. We Define:

\[ A_\varepsilon^c = \{ \omega \in \Omega \ s.t. \ |X_n(\omega) - \mu| \geq \varepsilon \} \subset \{ \omega \in \Omega \ s.t. \ X_n(\omega) - \mu \geq \varepsilon \} \cup \{ \omega \in \Omega \ s.t. \ \mu - X_n(\omega) \geq \varepsilon \} \]
We therefore have:

\[
\Pr\{\bar{X}_n - \mu \geq \epsilon\} = \Pr\{\sum_{i=1}^{n} X_n - n\mu \geq n\epsilon\} \leq e^{-s\epsilon} \mathbb{E}\left[e^{s\sum_{i=1}^{n} (X_i - n\mu)}\right]
\]

\[
= e^{-s\epsilon} \mathbb{E}\left[e^{s \sum_{i=1}^{n} (X_i - \mu)}\right]
\]

\[
= e^{-s\epsilon} \mathbb{E}\left[e^{s(X_1 - \mu)}\right] \mathbb{E}\left[e^{s(X_2 - \mu)}\right] \cdots \mathbb{E}\left[e^{s(X_n - \mu)}\right]
\]

\[
\leq e^{-s\epsilon} e^{n\frac{\epsilon^2}{2}} = e^{n\frac{\epsilon^2}{2} - s\epsilon}
\]

Since the above inequality holds for all values of \(s\), we set \(s = \frac{\epsilon}{c}\) to get \(e^{-\frac{n\epsilon^2}{2c}}\).

The same calculations can be repeated for \(\Pr\{\mu - \bar{X}_n \geq \epsilon\}\) and we therefore have:

\[
\Pr\{A_n^\epsilon\} \leq \Pr\{\bar{X}_n - \mu \geq \epsilon\} + \Pr\{\mu - \bar{X}_n \geq \epsilon\} \leq 2e^{-\frac{n\epsilon^2}{2c}}
\]

Which means that \(\sum_{n=0}^{\infty} \Pr\{A_n^\epsilon\} < \infty\) which concludes that \(A_n^\epsilon\) is not occurring infinitely often and therefore we have almost sure convergence for \(\bar{X}_n\) to \(\mu\).