**ECE 534 RANDOM PROCESSES** **FALL 2012**
**SOLUTIONS FOR SPROBLEM SET 2**

## 1 Stochastic convergence of sample variance

(a) We want to prove that $\mathbb{E}[\overline{V}_n] = \sigma^2$.

$$
\begin{aligned}
\mathbb{E}[\overline{V}_n] =& \frac{1}{n-1}\mathbb{E}[\sum_{i=1}^{n}(X_i - \overline{X}_n)^2] \\
=& \frac{1}{n-1}\mathbb{E}[\sum_{i=1}^{n}(X_i - \mu + \mu - \overline{X}_n)^2] \\
=& \frac{1}{n-1}\mathbb{E}[\sum_{i=1}^{n}(X_i - \mu)^2 + (\overline{X}_n - \mu)^2 - 2(X_i - \mu)(\overline{X}_n - \mu)] \\
=& \frac{1}{n-1}\sum_{i=1}^{n}\mathbb{E}[(X_i - \mu)^2] + \frac{1}{n-1}\sum_{i=1}^{n}\mathbb{E}[(\overline{X}_n - \mu)^2] - \frac{2}{n-1}\sum_{i=1}^{n}\mathbb{E}[(X_i - \mu)(\overline{X}_n - \mu)] \\
=& \frac{1}{n-1}[n\sigma^2] + \frac{1}{n-1}\sum_{i=1}^{n}\mathbb{E}[\left(\frac{\sum_{j=1}^{n} X_j - n\mu}{n}\right)^2] - \frac{2}{n-1}\sum_{i=1}^{n}\mathbb{E}[(X_i - \mu)(\frac{\sum_{j=1}^{n} X_j - n\mu}{n})] \\
=& \frac{1}{n-1}[n\sigma^2] + \frac{1}{(n-1)n^2}\sum_{i=1}^{n}\mathbb{E}[\left(\sum_{j=1}^{n}(X_j - \mu)\right)^2] - \frac{2}{(n-1)n}\sum_{i=1}^{n}\mathbb{E}[(X_i - \mu)\left(\sum_{j=1}^{n}(X_j - \mu)\right)] \\
=& \frac{1}{n-1}[n\sigma^2] + \frac{1}{(n-1)n^2}\sum_{i=1}^{n}\mathbb{E}[\sum_{j=1}^{n}(X_j - \mu)^2] - \frac{2}{(n-1)n}\sum_{i=1}^{n}\mathbb{E}[(X_i - \mu)\left(\sum_{j=1}^{n}(X_j - \mu)\right)] \\
=& \frac{1}{n-1}[n\sigma^2] + \frac{1}{(n-1)n^2}[n^2\sigma^2] - \frac{2}{(n-1)n}[n\sigma^2] \\
=& \frac{n}{n-1}[\sigma^2] + \frac{1}{(n-1)}[\sigma^2] - \frac{2}{(n-1)}[\sigma^2] = \sigma^2
\end{aligned}
$$

(b) In order to show how it converges to the same value in probability sense:

$$\begin{aligned}
\overline{V}_n =& \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X}_n)^2 \\
=& \frac{1}{n-1}\left(\sum_{i=1}^{n} X_i^2 + \sum_{i=1}^{n} \overline{X}_n^2 - 2\sum_{i=1}^{n} X_i \overline{X}_n\right) \\
=& \frac{1}{n-1}\left(\sum_{i=1}^{n} X_i^2 + n\overline{X}_n^2 - 2n\overline{X}_n^2\right) \\
=& \frac{n}{n-1}\frac{\sum_{i=1}^{n} X_i^2}{n} - \frac{n}{n-1}\left(\frac{\sum_{i=1}^{n} X_i}{n}\right)^2
\end{aligned}$$

Hereafter, it is straightforward. According to S.L.L.N., we know that $\frac{\sum_{i=1}^{n} X_i^2}{n} \xrightarrow{\text{a.s.}} \sigma^2 + \mu^2$ and $\frac{\sum_{i=1}^{n} X_i}{n} \xrightarrow{\text{a.s.}} \mu$ which conclude convergence in probability.

By writing the definition of convergence in probability, we can see that if $a_n \xrightarrow{\text{p.}} a$ and $b_n \xrightarrow{\text{p.}} b$ then first, $a_n^2 \xrightarrow{\text{p.}} a^2$ and second, $a_n + b_n \xrightarrow{\text{p.}} a + b$. Moreover, since $\frac{n}{n-1} \to 1$, we have $\overline{V}_n \xrightarrow{\text{p.}} \sigma^2 + \mu^2 - \mu^2 = \sigma^2$.

## 2 Wild oscillations

First, we review the convergense in probability sense. For every $\epsilon > 0$ we have $\mathbb{P}\{|X_n - 0| > \epsilon\} = \frac{1}{n^2}$. It can be easily concluded that $\lim_{n\to\infty} \mathbb{P}\{|X_n - 0| > \epsilon\} = 0$. Therefore, $X_n \xrightarrow{\text{p.}} 0$.

Convergense in distribution can also be concluded from the convergence in probability.

For considering convergence in *almost sure* sense, note that the probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$ is not given to us. Therefore, we cannot directly apply the definition of almost sure convergence. However, we can use Borell-Cantelli lemma to evaluate it:

For each $1 > \epsilon > 0$ Define:

$$A_n^\epsilon = \{\omega \in \Omega \ s.t. \ |X_n(\omega)| > \epsilon\}$$

We want to prove that

$$\mathbb{P}(\cap_{m=0}^{\infty} \cup_{n=m}^{\infty} A_n^\epsilon) = 0$$

Which is equivalent to the definition of the convergence in almost sure sense. Acording to Borell-Cantelli lemma, we only need to consider that:

$$\sum_{n=0}^{\infty} \mathbb{P}(A_n^\epsilon) = \sum_{n=0}^{[\frac{1}{\epsilon}]} 1 + \sum_{n=[\frac{1}{\epsilon}]}^{\infty} \frac{1}{n^2} < \infty$$

For convergence in mean square sense, we re-write the definition:

$$\begin{aligned}\mathbb{E}[(X_n - 0)^2] = \mathbb{E}[X_n^2] =& \frac{1}{n^2}\mathbb{P}\{X_n = \frac{1}{n}\} + n^2\mathbb{P}\{X_n = n\} \\ =& \frac{1}{n^2}(1 - \frac{1}{n^2}) + n^2\frac{1}{n^2} \\ =& 1 + \frac{1}{n^2} - \frac{1}{n^4}\end{aligned}$$

This means that

$$\lim_{n\to\infty} \mathbb{E}[(X_n - 0)^2] = 1 \neq 0$$

And we do not have convergence in mean-square sense.

## 3 Convergence to a constant in mean square sense

We first prove the "if" part. It's just enough to remember the definition of convergence in mean-square sense. We say $\{X_n\}$ converges to $X$ in mean square sense if $\mathbb{E}[X_n^2] < \infty$ for all $n$ and $\lim_{n\to\infty} \mathbb{E}[(X_n - X)^2] = 0$.

Since $\lim_{n\to\infty} \text{Var}[X_n]$ exists, we can conclude that it is bounded for all $n$. Therefore, $\mathbb{E}[X_n^2] = \text{Var}[X_n] + \mathbb{E}[X_n]^2$ is bounded.

We now take $X = c$ in the definition of mean square convergence:

$$\lim_{n\to\infty} \mathbb{E}[(X_n - c)^2] = \lim_{n\to\infty} \mathbb{E}[(X_n - \mathbb{E}[X_n])^2] = \lim_{n\to\infty} \text{Var}[X_n] = 0$$

For the "only if" part;

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}[(X_n - c)^2] &= \lim_{n\to\infty} \left[\mathbb{E}[X_n^2] + c^2 - 2c\mathbb{E}[X_n]\right] \\ &= \lim_{n\to\infty} \left[\mathbb{E}[X_n^2] - \mathbb{E}[X_n]^2 + \mathbb{E}[X_n]^2 + c^2 - 2c\mathbb{E}[X_n]\right] \\ &= \lim_{n\to\infty} \left[\text{Var}[X_n] + (\mathbb{E}[X_n] - c)^2\right] \\ &= \lim_{n\to\infty} \text{Var}[X_n] + \lim_{n\to\infty} (\mathbb{E}[X_n] - c)^2 = 0 \\ &\Rightarrow \begin{cases}\lim_{n\to\infty} \text{Var}[X_n] = 0 \\ \lim_{n\to\infty} (\mathbb{E}[X_n] - c)^2 = 0\end{cases}\end{aligned}$$

Where the last conclusion can be made because the only way for the sum of two non-negative terms to be zero is for both of them to be zero.

## 4 Stochastic convergence of a minimum

For this problem, we first need to find the probability distribution of $X_n = n\min\{Z_1, \cdots, Z_n\}$.

$$\begin{aligned}F_{X_n}(c) = \mathbb{P}\{X_n \leq c\} =& \mathbb{P}\{\min\{Z_1, \cdots, Z_n\} \leq \frac{c}{n}\} \\ =& 1 - \mathbb{P}\{\min\{Z_1, \cdots, Z_n\} > \frac{c}{n}\} \\ =& 1 - \mathbb{P}\{Z_1 > \frac{c}{n}\}\mathbb{P}\{Z_2 > \frac{c}{n}\}\cdots\mathbb{P}\{Z_n > \frac{c}{n}\} \\ =& 1 - \left(1 - F_Z(\frac{c}{n})\right)^n \\ =& 1 - \left(1 - \int_0^{\frac{c}{n}} f_Z(y)dy\right)^n\end{aligned}$$

Now, all we need to do is to prove that:

$$\lim_{n\to\infty} 1 - \left(1 - \int_0^{\frac{c}{n}} f_Z(y)dy\right)^n = 1 - \exp(-\lambda c)$$

We know that according to mean value theorem, there exist $x_n \in (0, \frac{c}{n})$ such that $\int_0^{\frac{c}{n}} f_Z(y)dy = f_Z(x_n)\frac{c}{n}$. Therefore, we have:

$$\lim_{n\to\infty}\left(1 - \int_0^{\frac{c}{n}} f_Z(y)dy\right)^n = \lim_{n\to\infty}\left(1 - \frac{c}{n}f_Z(x_n)dy\right)^n = \exp(-c\lim_{n\to\infty} f_Z(x_n)) = \exp(-\lambda c)$$

Where the last conclusion can be made because as $n$ goes to infinity, we have $\frac{c}{n}$ thus $x_n$ converging to zero form the right. We also know from the assumption that $\lim_{x\to 0^+} f(x) = \lambda$ so $\lim_{n\to\infty} f(x_n) = \lambda$.

## 5 Subgaussian random variables

(a) (i) $X$ is sugaussian when $X \sim \mathcal{N}(\mu, \sigma^2)$ :

$$\begin{aligned}\mathbb{E}[e^{t(X-\mathbb{E}X)}] =& \mathbb{E}[e^{tX}e^{-t\mathbb{E}X}] \\ =& \mathbb{E}[e^{tX}]e^{-t\mu} \\ =& \Phi_X(-jt)e^{-t\mu} \\ =& e^{t\mu + \frac{t^2\sigma^2}{2}}e^{-t\mu} = e^{\frac{t^2\sigma^2}{2}} \leq e^{\frac{ct^2}{2}} \Leftrightarrow \sigma^2 \leq c\end{aligned}$$

(a) (ii) It is subgaussian. Note that $\mathbb{E}X = 0$ thus $\mathbb{E}[e^{t(X-\mathbb{E}X)}] = \frac{e^{tM}+e^{-tM}}{2}$. Considering fourier expansion for $e^{tM}$ we have:

$$
\begin{aligned}
e^{tM} &= 1 + \frac{tM}{1!} + \frac{(tM)^2}{2!} + \frac{(tM)^3}{3!} + \cdots \\
e^{-tM} &= 1 - \frac{tM}{1!} + \frac{(tM)^2}{2!} - \frac{(tM)^3}{3!} + \cdots \\
\Rightarrow \frac{e^{tM} + e^{-tM}}{2} &= 1 + \frac{(tM)^2}{2!} + \frac{(tM)^4}{4!} + \cdots \\
\leq 1 + \frac{(tM)^2}{1!} + \frac{(tM)^4}{2!} &+ \cdots = e^{t^2M^2} \leq e^{\frac{ct^2}{2}} \Leftrightarrow M^2 \leq \frac{c}{2}
\end{aligned}
$$

(a) (iii) It is not subgaussian. Note that $\mathbb{E}X = \lambda$ when we have exponential distribution.

$$
\begin{aligned}
\mathbb{E}[e^{t(X-\mathbb{E}X)}] =& \mathbb{E}[e^{tX} e^{-t\mathbb{E}X}] \\
=& \mathbb{E}[e^{tX}] e^{-t\lambda} \\
=& \left[ \int_0^\infty e^{tx} \ \lambda e^{-\lambda x} dx \right] e^{-t\lambda} \\
=& e^{-t\lambda} \int_0^\infty \lambda e^{(t-\lambda)x} dx
\end{aligned}
$$

The above integral diverges for $t > \lambda$ and cannot be bounded by any function such as $e^{\frac{ct^2}{2}}$ for all values of $t$. Therefore, this r.v. is not subgaussian.

(b) First of all, note that $\mathbb{E}Y = a_1 \mathbb{E}X_1 + \cdots + a_n \mathbb{E}X_n$. We therefore have:

$$
\begin{aligned}
\mathbb{E}[e^{t(Y-\mathbb{E}Y)}] =& \mathbb{E}[e^{tY}] e^{-t\mathbb{E}Y} \\
=& \mathbb{E}[e^{t(a_1X_1 + \cdots + a_nX_n)}] e^{-t\mathbb{E}Y} \\
=& \mathbb{E}[e^{t(a_1X_1)}] \cdots \mathbb{E}[e^{t(a_nX_n)}] e^{-ta_1\mathbb{E}X_1} \cdots e^{-ta_n\mathbb{E}X_n} \\
=& \mathbb{E}[e^{ta_1(X_1-\mathbb{E}X_1)}] \cdots \mathbb{E}[e^{ta_n(X_n-\mathbb{E}X_n)}] \\
\leq& e^{\frac{c(a_1t)^2}{2}} \cdots e^{\frac{c(a_nt)^2}{2}} \\
=& e^{\frac{ct^2}{2}(a_1^2 + \cdots a_n^2)} = e^{\frac{ct^2}{2}}
\end{aligned}
$$

(c) Again, like question 2, we use Borell-Cantelli Lemma for the proof of almost sure convergence. We Define:

$$
A_n^\epsilon = \{\omega \in \Omega \ s.t. \ |\bar{X}_n(\omega) - \mu| \geq \epsilon\} \subset \{\omega \in \Omega \ s.t. \ \bar{X}_n(\omega) - \mu \geq \epsilon\} \cup \{\omega \in \Omega \ s.t. \ \mu - \bar{X}_n(\omega) \geq \epsilon\}
$$

We therefore have:

$$\begin{aligned}\mathbb{P}\{\bar{X}_n - \mu \geq \epsilon\} = \mathbb{P}\{\sum_{i=1}^{n} X_n - n\mu \geq n\epsilon\} \leq \ & e^{-sn\epsilon}\mathbb{E}[e^{s(\sum_{i=1}^{n} X_i - n\mu)}] \\ =& e^{-sn\epsilon}\mathbb{E}[e^{s\sum_{i=1}^{n}(X_i-\mu)}] \\ =& e^{-sn\epsilon}E[e^{s(X_1-\mu)}]E[e^{s(X_2-\mu)}]\cdots E[e^{s(X_n-\mu)}] \\ \leq& e^{-sn\epsilon}e^{\frac{ncs^2}{2}} = e^{n(\frac{cs^2}{2}-s\epsilon)}\end{aligned}$$

Since the above inequality holds for all values of $s$, we set $s = \frac{\epsilon}{c}$ to get $e^{\frac{-n\epsilon^2}{2c}}$.

The same calculations can be repeated for $\mathbb{P}\{\mu - \bar{X}_n \geq \epsilon\}$ and we therefore have:

$$\mathbb{P}\{A_n^{\epsilon}\} \leq \mathbb{P}\{\bar{X}_n - \mu \geq \epsilon\} + \mathbb{P}\{\mu - \bar{X}_n \geq \epsilon\} \leq 2e^{\frac{-n\epsilon^2}{2c}}$$

Which means that $\sum_{n=0}^{\infty} \mathbb{P}\{A_n^{\epsilon}\} < \infty$ which concludes that $A_n^{\epsilon}$ is not occuring infinitly often and therefore we have almost sure convergence for $\bar{X}_n$ to $\mu$.