ECE 534 RANDOM PROCESSES **FALL 2012**
SOLUTIONS TO PROBLEM SET 1

1. Persistence and coin tossing

(a) If 0 denotes `T` and 1 denotes `H`, then Ω consists of all finite binary strings of length at least two, which end in 11 and otherwise consist of 0's and 10's in various combinations (any number of consecutive tosses that come up `T` are allowed, but if the coin comes up `H` any time before the last three tosses, then the next toss must give `T`).

(b) Let p_k denote the probability that k tosses (including the two heads at the end) suffice. Because at least 2 tosses are needed, we have $p_0 = p_1 = 0$. Because the coin tosses are independent, we have $p_2 = 1/4$ and $p_3 = 1/8$ (the only legal string with $k = 3$ is 011). For $k \geq 4$, let n_k denote the number of all legal strings of length k. By independence and the fact that the coin is fair,

$$p_k = \frac{n_k}{2^k}.$$

A legal sequence of tosses can start either with 0 or with 10. Moreover, the string that follows either the initial 0 or the initial 10 must be a legal string of length $k-1$ or $k-2$ respectively. Therefore,

$$\begin{aligned} n_k &= \#(\text{legal strings that start with 0}) + \#(\text{legal strings that start with 10}) \\ &= n_{k-1} + n_{k-2}, \qquad k = 4, 5, \ldots \end{aligned}$$

With the initial conditions $n_2 = 1$ and $n_3 = 1$, we have $n_k = F_{k-1}$, where F_i denotes the ith Fibonacci number (recall that the Fibonacci sequence $F_0, F_1, F_2, \ldots$ is defined recursively as $F_0 = 0, F_1 = 1, F_i = F_{i-1} + F_{i-2}$ for $i \geq 2$). Thus,

$$p_k = \frac{F_{k-1}}{2^k}$$

2. Independence and extreme events

Suppose that $P(E) = 0$. For any other event A, $E \cap A \subseteq E$, so by the monotonicity property of probability we have $P(E \cap A) \leq P(E) = 0$. On the other hand, $P(A \cap E) \geq 0$, so $P(E \cap A) = 0 = P(E)P(A)$, where the last equality is due to the fact that $P(E) = 1$. Hence, any E with $P(E) = 0$ is independent of all other events.

Now suppose $P(E) = 1$. By the law of total probability, $P(E \cap A) = P(A) - P(E^c \cap A)$. Since $P(E^c) = 0$, E^c and A are independent (as we have just proved), so $P(E^c \cap A) = 0$. Therefore, $P(E \cap A) = P(A) = P(E)P(A)$, where the last equality is due to the fact that $P(E) = 1$. Hence, any E with $P(E) = 1$ is independent of all other events.

3. Some conditional probabilities

(a) False. Choose any three events A, B, C such that $A \cap B = C$ and $P(C) < 1$. Then $P(A \cap B|C) = P(A \cap B \cap C)/P(C) = 1$, but $P(A \cap B) = P(C) \neq 1$.

(b) True. By definition of conditional probability, $P(C) > 0$ and $P(A \cap B \cap C) = P(A \cap B|C)P(C) = P(C)$.

(c) True. $P(A^c|C) = 1 - P(A|C) \leq 1 - P(A \cap B|C) = 0$. Since probabilities are nonnegative, it must be the case that $P(A^c|C) = 0$.

(d) False. $P(A \cap B|C) = 1$ for any C with $0 < P(C) < 1$ and $C \subseteq A \cap B$. Then $A \cap B \cap C = C$, so $P(A \cap B \cap C) = P(C)$.

4. Some more conditional probabilities.

Since $P(B) > 0$ and $P_B(A) = P(A \cap B)/P(B) \geq 0$ for any $A \in \mathcal{F}$, Axiom **P.1** is satisfied. Now let $A_1, A_2 \in \mathcal{F}$ be mutually exclusive (i.e., $A_1 \cap A_2 = \varnothing$). Then $(A_1 \cap B) \cup (A_2 \cap B) = \varnothing$ as well, therefore $A_1 \cap B$ and $A_2 \cap B$ are mutually exclusive. Consequently,

$$\begin{aligned} P_B(A_1 \cup A_2) &= \frac{P\big((A_1 \cup A_2) \cap B\big)}{P(B)} \\ &= \frac{P\big((A_1 \cap B) \cup (A_2 \cap B)\big)}{P(B)} \\ &= \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)} \\ &= P_B(A_1) + P_B(A_2). \end{aligned}$$

The same argument applies to any countably infinite sequence $A_1, A_2, \ldots$ of mutually exclusive events. So Axiom **P.2** is satisfied. Finally, $P_B(\Omega) = P(\Omega \cap B)/P(B) = P(B)/P(B) = 1$, so Axiom **P.3** is satisfied also. In view of all this, P_B is a bona fide probability measure on $(\Omega, \mathcal{F})$.

5. Conditional independence.

(a) Applying the definition of conditional probability repeatedly, we can write

$$\begin{aligned} P(A \cap B \cap C) &= P(B)P(A \cap C|B) \\ &= P(B)P(A|B)P(C|B) \\ &= P(A \cap B)P(C|B) \\ &= P(A)P(B|A)P(C|B). \end{aligned}$$

The proof of the second equality is similar.

(b) By definition of conditional probability,

$$\begin{aligned} P(A|B, C) &\equiv P(A|B \cap C) \\ &= \frac{P(A \cap B \cap C)}{P(B \cap C)} \\ &= \frac{P(B)P(A \cap C|B)}{P(B)P(C|B)} \\ &= P(A|B). \end{aligned}$$

(c) False. For A and B to be conditionally independent given C, we must have

$$P(A \cap B \cap C) = P(A|C)P(B|C)P(C). \tag{1}$$

Since A and C are conditionally independent given B by hypothesis, we also have

$$P(A \cap B \cap C) = P(C)P(A|B)P(B|C). \tag{2}$$

Suppose that B and C are such that $P(B|C) > 0$. Then if (1) and (2) both hold, we must have $P(A|C) = P(A|B)$, which is the same as $P(A \cap C)/P(A \cap B) = P(C)/P(B)$. This is not true in general (although it may be true for specific choices of A, B, C.)

6. Partitions

By the law of total probability and the fact that $P(A) > 0$, we have

$$\begin{aligned}
P(A) &= \sum_{i=1}^{k} P(E_i \cap A) \\
&= \sum_{i=1}^{k} P(E_i|A)P(A) \\
&= P(E_1|A)P(A) + \sum_{i=2}^{k} P(E_i|A)P(A) \\
&< P(E_1)P(A) + \sum_{i=2}^{k} P(E_i|A)P(A) \\
&= \left(P(E_1) + \sum_{i=2}^{k} P(E_i|A) \right) P(A).
\end{aligned}$$

Suppose that $P(E_i|A) \leq P(E_i)$ for all $i \in \{2, \ldots, k\}$. Then

$$P(A) < \left(\sum_{i=1}^{k} P(E_i) \right) P(A) = P(A),$$

where the last step is because $E_1, \ldots, E_k$ partition Ω. But $P(A) < P(A)$ is a contradiction, so there must be at least one $i \in \{2, \ldots, k\}$ such that $P(E_i|A) > P(E_i)$.

7. Nonlinear transformation of a random variable

(a) Since X takes values in $[-1, 1]$, Y can only take values in $[-1, 1]$ – in fact, if $X \in [-1, 0]$, then $Y \in [-1, 0]$, but if $X \in (0, 1]$, then $Y = 1$. Hence, $Y \in [-1, 0] \cup \{1\}$. For $y \in [-1, 0]$,

$$F_Y(y) = P(Y \leq y) = P(X \leq y) = F_X(y) = \frac{1}{2}(y + 1).$$

For $y \in (0, 1)$,

$$\begin{aligned}
F_Y(y) &= P(\{Y \leq 0\} \cup \{0 < Y < 1\}) \\
&= P(Y \leq 0) + P(0 < Y < 1) \\
&= P(Y \leq 0) \\
&= P(X \leq 0) \\
&= \frac{1}{2}.
\end{aligned}$$

Finally, for $Y \geq 1$, $F_Y(y) = 1$. Therefore,

$$F_Y(y) = \begin{cases} 0, & y < -1 \\ \frac{1}{2}(y+1), & -1 \leq y < 0 \\ \frac{1}{2}, & 0 \leq y < 1 \\ 1, & y \geq 1 \end{cases} \tag{3}$$

The graph of the cdf is sketched in Figure 1.

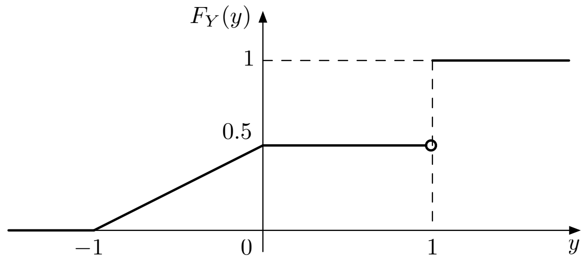

Figure 1: The cdf of Y from Problem 7.

(b) Either by differentiating (3) using generalized derivatives, or by inspection,

$$f_Y(y) = \frac{1}{2} I_{[-1,0]}(y) + \frac{1}{2}\delta(y-1),$$

where I_A denotes the indicator function of the set A.

(c) For the expectation of Y, we have

$$\begin{aligned}
\mathbb{E}[Y] &= \int_{-\infty}^{\infty} y f_Y(y) \mathrm{d}y \\
&= \frac{1}{2}\int_{-1}^{0} y \, \mathrm{d}y + \frac{1}{2}\int_{-\infty}^{\infty} y\delta(y-1) \, \mathrm{d}y \\
&= -\frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot 1 \\
&= \frac{1}{4}
\end{aligned}$$

To compute the variance of Y, we first find $\mathbb{E}[Y^2]$:

$$\begin{aligned}
\mathbb{E}[Y^2] &= \int_{-\infty}^{\infty} y^2 f_Y(y) \mathrm{d}y \\
&= \frac{1}{2}\int_{-1}^{0} y^2 \, \mathrm{d}y + \frac{1}{2}\int_{-\infty}^{\infty} y^2\delta(y-1) \, \mathrm{d}y \\
&= \frac{1}{2}\cdot\frac{1}{3} + \frac{1}{2}\cdot 1 \\
&= \frac{2}{3}.
\end{aligned}$$

Thus, $\mathrm{Var}(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = \frac{2}{3} - \frac{1}{16} = \frac{29}{48}$.

(d) Since Y can only take values in $[-1,0] \cup \{1\}$, the conditional expectation $\mathbb{E}[X|Y = y]$ is defined only for $y \in [-1,0]$ and for $y = 1$. When $Y \in [-1,0]$, $X = Y$ by definition of the function mapping X to Y.

Therefore, $\mathbb{E}[X|Y=y]=y$ for $y\in[-1,0]$. On the other hand, $Y=1$ if and only if $X\in(0,1]$, and so

$$\begin{aligned}\mathbb{E}[X|Y=1] &= \int_{-\infty}^{\infty} x\,\mathrm{d}P_{X|Y}(x|1)\\ &= \int_{-\infty}^{\infty} x\frac{P(X\le x, Y=1)}{P(Y=1)}\mathrm{d}x\\ &= \int_0^1 x\,\mathrm{d}x\\ &= \frac{1}{2}\end{aligned}$$

8. Functions of a Gaussian random variable

(a) The characteristic function of a $N(0,\sigma^2)$ random variable X is

$$\Phi(u)=\mathbb{E}\left[e^{juX}\right]=\exp\left(-\frac{u^2\sigma^2}{2}\right).$$

Using the Euler formula $e^{ju}=\cos u + j\sin u$ and the definition of the characteristic function, we have

$$\begin{aligned}\mathbb{E}[\cos(nX)] &= \frac{\mathbb{E}\left[e^{jnX}\right]+\mathbb{E}\left[e^{-jnX}\right]}{2}\\ &= \frac{\Phi(n)+\Phi(-n)}{2}\\ &= \exp\left(-\frac{n^2\sigma^2}{2}\right)\end{aligned}$$

(b) The characteristic function of a $N(0,\sigma^2)$ is everywhere infinitely differentiable. From the lecture notes,

$$\mathbb{E}[X^n]=\frac{\Phi^{(n)}(0)}{j^n}.$$

To read off the derivatives of Φ at 0, we use the Maclaurin series for the exponential function, which is everywhere convergent:

$$e^u=\sum_{n=0}^{\infty}\frac{u^n}{n!}$$

Therefore,

$$\Phi(u)=\exp\left(-\frac{u^2\sigma^2}{2}\right)=\sum_{n=0}^{\infty}\frac{(-1)^n\sigma^{2n}u^{2n}}{2^n n!}=\sum_{n\text{ odd}}\frac{0}{n!}u^n+\sum_{n\text{ even}}\frac{(-1)^{n/2}\sigma^n}{2^{n/2}(n/2)!}u^n$$

Since the coefficient of u^n in the Maclaurin series of a function f is equal to $f^{(n)}(0)/n!$, we have

$$\Phi^{(n)}(0)=\begin{cases}0, & \text{if } n \text{ is odd}\\ \frac{(-1)^{n/2}n!\sigma^n}{2^{n/2}(n/2)!}, & \text{if } n \text{ is even}\end{cases}$$

Since $j^n=(-1)^{n/2}$ when n is even, we have

$$\mathbb{E}[X^n]=\begin{cases}0, & \text{if } n \text{ is odd}\\ \frac{\sigma^n}{2^{n/2}}\frac{n!}{(n/2)!}, & \text{if } n \text{ is even}\end{cases}$$

If you are familiar with the Euler Gamma function, then you will recognize that, for n even, $\mathbb{E}[X^n] = \sigma^n(n-1)\cdot(n-3)\cdot\ldots\cdot 3\cdot 1$.

9. Two friends on a hot summer day

(a) Let 0 and 1 denote `T` and `H`, respectively. Then

$$\begin{aligned} P(Y \le y) &= P(Y \le y, X=0) + P(Y \le y, X=1) \\ &= P(Y \le y | X=0)P(X=0) + P(Y \le y | X=1)P(X=1) \\ &= \frac{1}{2}P(Y \le y|X=0) + \frac{1}{2}P(Y \le y|X=1). \end{aligned}$$

From the problem formulation, $P(Y \le y|X=0) = u(y)$ and $P(Y \le y|X=1) = (1-e^{-y})u(y)$, where $u(y)$ is the unit step function. Hence,

$$F_Y(y) = \frac{1}{2}u(y) + \frac{1}{2}(1-e^{-u})u(y) = \left(1 - \frac{1}{2}e^{-u}\right)u(y). \tag{4}$$

The graph of the cdf of Y is shown in Figure 2. (b) Differentiating (4) using generalized derivatives, we get

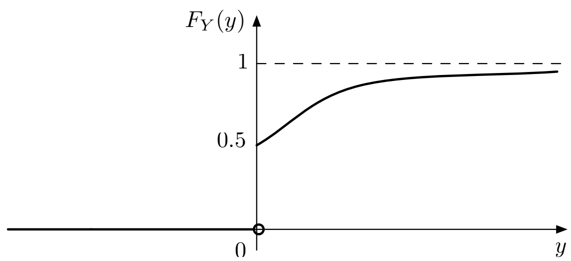

Figure 2: The cdf of Y from Problem 9.

$$f_Y(y) = \frac{1}{2}\delta(y) + \frac{1}{2}e^{-y}u(y) \tag{5}$$

(c) Using (5),

$$\mathbb{E}[Y] = \int_{-\infty}^{\infty} y f_Y(y)\,\mathrm{d}y = \frac{1}{2}\underbrace{\int_{-\infty}^{\infty} y\delta(y)\,\mathrm{d}y}_{=0} + \frac{1}{2}\underbrace{\int_0^{\infty} ye^{-y}\mathrm{d}y}_{=1} = \frac{1}{2}$$

Now let us compute $\mathbb{E}[Y^2]$:

$$\mathbb{E}[Y^2] = \frac{1}{2}\underbrace{\int_{-\infty}^{\infty} y^2\delta(y)\,\mathrm{d}y}_{=0} + \frac{1}{2}\underbrace{\int_0^{\infty} y^2e^{-y}\,\mathrm{d}y}_{=2} = 1$$

Thus, $\mathrm{Var}(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = 1 - 1/4 = 3/4$.

A quicker way of finding $\mathbb{E}[Y]$ and $\mathrm{Var}(Y)$ is to note that the distribution of Y is a mixture of two distributions: the distribution of a random variable Y_0 equal to 0 with probability 1 and the distribution of an Exp(1) random variable Y_1, each entering with weight 1/2. By linearity of expectation,

$$\mathbb{E}[Y] = \frac{1}{2}\mathbb{E}[Y_0] + \frac{1}{2}\mathbb{E}[Y_1] = \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 1 = \frac{1}{2};$$
$$\mathbb{E}[Y^2] = \frac{1}{2}\mathbb{E}[Y_0^2] + \frac{1}{2}\mathbb{E}[Y_1^2] = \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 2 = 1.$$

From the second line, $\mathrm{Var}(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = 1 - 1/4 = 3/4$.

(d) The desired probability is

$$P(Y < 3) = P(Y \le 3) = 1 - \frac{1}{2}e^{-3}.$$

10. The magic of Poissonization

(a) Since X and Y must satisfy $X + Y = n$, they are certainly dependent. More formally, the joint pmf p_{XY} is

$$p_{XY}(x, y) = \binom{n}{x} p^x (1-p)^{n-x} 1_{\{x+y=n\}},$$

whereas the pmfs p_X and p_Y are

$$p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}, \qquad p_Y(y) = \binom{n}{y} p^{n-y} (1-p)^y$$

From this it is easy to see that $p_X Y(x, y) \neq p_X(x) p_Y(y)$.

(b) The joint pmf of X, Y and N is

$$\begin{aligned} p_{XYN}(x, y, n) &= p_{XY|N}(x, y|n) p_N(n) \\ &= \binom{n}{x} p^x (1-p)^{n-x} 1_{\{x+y=n\}} \cdot \frac{\lambda^n e^{-\lambda}}{n!}. \end{aligned}$$

To determine p_{XY}, we compute the marginal pmf:

$$\begin{aligned} p_{XY}(x, y) &= \sum_{n=0}^{\infty} p_{XYN}(x, y, n) \\ &= \sum_{n=0}^{\infty} \binom{n}{x} p^x (1-p)^{n-x} 1_{\{x+y=n\}} \cdot \frac{\lambda^n e^{-\lambda}}{n!} \\ &= \binom{x+y}{x} p^x (1-p)^y \frac{\lambda^{x+y} e^{-\lambda}}{(x+y)!} \\ &= \frac{(x+y)!}{x! y!} p^x (1-p)^y \frac{\lambda^{x+y} e^{-\lambda}}{(x+y)!}. \end{aligned}$$

Collecting terms involving x and those involving y, we get

$$\begin{aligned} p_{XY}(x, y) &= \frac{(\lambda p)^x}{x!} \frac{((1-p)\lambda)^y}{y!} e^{-\lambda} \\ &= \frac{(\lambda p)^x e^{-\lambda p}}{x!} \cdot \frac{(\lambda(1-p))^y e^{-\lambda(1-p)}}{y!}, \end{aligned}$$

which is a product of two pmf's, one of a Poisson(λp) r.v., and another of Poisson($\lambda(1-p)$) r.v.. In fact, $X \sim$ Poisson(λp) and $Y \sim$ Poisson($\lambda(1-p)$). Hence, $p_{XY}(x, y) = p_X(x) p_Y(y)$, so X and Y are independent.