5. Markov Processes and All That

Assigned Reading: Chapter 4 and Sections 1–3 of Chapter 6 of the course notes.

Problems to be handed in:

Problems 4.31 and 4.37 from the course notes, as well as the following three problems:

1. A compound process
Let $X = (X_n)_{n=1}^\infty$ be an i.i.d. binary random process with $P(X_n = \pm 1) = 1/2$, and let $N = (N_t)_{t\geq 0}$ be a Poisson process with rate $\lambda > 0$. Assume that $X$ and $N$ are independent of one another. Let us define another continuous-time process $Y = (Y_t)_{t\geq 0}$ by

$$Y_t = \begin{cases} 0, & t = 0 \\ \sum_{i=1}^{N_t} X_i, & t > 0 \end{cases}$$

(a) Find the expectation $\mu_Y(t)$ and the covariance function $C_Y(s,t)$.

(b) Does $Y$ have independent increments? Justify your answer.

(c) Is $Y$ a martingale? Justify your answer.

2. Random walk on the $N$-cycle
Let $S = \{0, 1, \ldots, N - 1\}$. Consider a discrete-time time-homogeneous Markov process $X = (X_n)_{n=0}^\infty$ with one-step transition matrix $P = (p_{ij} : i, j \in S)$ that has entries

$$p_{ij} = \begin{cases} 1/2, & \text{if } j \equiv i + 1 \mod N \\ 1/2, & \text{if } j \equiv i - 1 \mod N \\ 0, & \text{otherwise} \end{cases}$$

(a) Show that $X$ admits a recursive representation of the form $X_n = f(X_{n-1}, U_n)$, where $U = (U_n)$ is an i.i.d. binary process with $P(U_n = \pm 1) = 1/2$.

(b) Show that $X$ is irreducible.

(c) Show that $X$ is periodic if $N$ is even and aperiodic if $N$ is odd.

(d) If $N$ is odd, find the smallest value of $r \in \mathbb{N}$ such that $p_{ij}^r > 0$ for all $i, j \in S$.

3. Random walk on a simple undirected graph
A simple undirected graph is a pair $G = (V, E)$, where $V$ is a finite vertex set and $E$ is an edge set consisting of unordered pairs $(i, j) \in V$ with $i \neq j$. If $(i, j) \in E$, which automatically means that $(j, i) \in E$ as well, we say that $G$ has an edge between $i$ and $j$. The degree of a vertex $i \in V$, denoted by $\deg(i)$, is the number of vertices $j \in V$ such that $(i, j) \in E$. The simple random walk on $G$ is a discrete-time time-homogeneous Markov process $X = (X_n)$, whose transition matrix has entries

$$p_{ij} = \begin{cases} \frac{1}{\deg(i)}, & \text{if } (i, j) \in E \\ 0, & \text{otherwise} \end{cases}$$
(a) Show that $\pi$ with $\pi_i = \text{deg}(i)/2|E|$ for all $i \in V$ is a stationary distribution for this Markov process.

(b) We say that the graph $G$ is \textit{connected} if for any two vertices $i, j \in V$ there exists a sequence of vertices $i_0, i_1, \ldots, i_k$ such that $i_0 = i$, $i_k = j$, and $(i_\ell, i_{\ell+1}) \in E$ for all $\ell = 0, \ldots, k - 1$. Show that the simple random walk on $G$ is irreducible if and only if $G$ is connected.