ECE 534 RANDOM PROCESSES **FALL 2012**
PROBLEM SET 5 **Due beginning of class, Thursday, November 29**

5. Markov Processes and All That

Assigned Reading: Chapter 4 and Sections 1–3 of Chapter 6 of the course notes.

Problems to be handed in:

Problems 4.31 and 4.37 from the course notes, as well as the following three problems:

1 A compound process

Let $X = (X_n)_{n=1}^{\infty}$ be an i.i.d. binary random process with $P(X_n = \pm 1) = 1/2$, and let $N = (N_t)_{t \ge 0}$ be a Poisson process with rate $\lambda > 0$. Assume that X and N are independent of one another. Let us define another continuous-time process $Y = (Y_t)_{t \ge 0}$ by

$$Y_t = \begin{cases} 0, & t = 0 \\ \sum_{i=1}^{N_t} X_i, & t > 0 \end{cases}$$

(a) Find the expectation $\mu_Y(t)$ and the covariance function $C_Y(s,t)$.

(b) Does Y have independent increments? Justify your answer.

(c) Is Y a martingale? Justify your answer.

2 Random walk on the N-cycle

Let $\mathcal{S} = \{0, 1, \dots, N-1\}$. Consider a discrete-time time-homogeneous Markov process $X = (X_n)_{n=0}^{\infty}$ with one-step transition matrix $P = (p_{ij} : i, j \in \mathcal{S})$ that has entries

$$p_{ij} = \begin{cases} 1/2, & \text{if } j \equiv i+1 \mod N \\ 1/2, & \text{if } j \equiv i-1 \mod N \\ 0, & \text{otherwise} \end{cases}$$

(a) Show that X admits a recursive representation of the form $X_n = f(X_{n-1}, U_n)$, where $U = (U_n)$ is an i.i.d. binary process with $P(U_n = \pm 1) = 1/2$.

(b) Show that X is irreducible.

(c) Show that X is periodic if N is even and aperiodic if N is odd.

(d) If N is odd, find the smallest value of $r \in \mathbb{N}$ such that $p_{ij}^r > 0$ for all $i, j \in \mathcal{S}$.

3 Random walk on a simple undirected graph

A *simple undirected graph* is a pair $G = (V, E)$, where V is a finite *vertex set* and E is an *edge set* consisting of unordered pairs $(i,j) \in V$ with $i \neq j$. If $(i,j) \in E$, which automatically means that $(j,i) \in E$ as well, we say that G has an *edge* between i and j. The *degree* of a vertex $i \in V$, denoted by $\deg(i)$, is the number of vertices $j \in V$ such that $(i,j) \in E$. The *simple random walk on G* is a discrete-time time-homogeneous Markov process $X = (X_n)$, whose transition matrix has entries

$$p_{ij} = \begin{cases} \frac{1}{\deg(i)}, & \text{if } (i,j) \in E \\ 0, & \text{otherwise} \end{cases}$$

(a) Show that π with $\pi_i = \deg(i)/2|E|$ for all $i \in V$ is a stationary distribution for this Markov process.

(b) We say that the graph G is *connected* if for any two vertices $i, j \in V$ there exists a sequence of vertices $i_0, i_1, \dots, i_k$ such that $i_0 = i$, $i_k = j$, and $(i_\ell, i_{\ell+1}) \in E$ for all $\ell = 0, \dots, k-1$. Show that the simple random walk on G is irreducible if and only if G is connected.