3. MMSE Estimation

Assigned Reading: Chapter 3 of the course notes.

Reminders: The first midterm exam will be on Wednesday, October 24, from 6 pm until 7:30 pm in 165 Everitt Lab.

Problems to be handed in:

Problems 3.1 and 3.5 from the course notes, as well as the following three problems:

1 Estimation of a symmetric binary signal in Gaussian noise
Let $X$ take values in the set $\{-1, +1\}$, where each of the two possible values has probability $1/2$. Consider the problem of estimating $X$ on the basis of an observation $Y = X + \sqrt{\rho}Z$, where $\rho > 0$ is a deterministic parameter that quantifies the signal-to-noise ratio, and the additive noise $Z \sim N(0, 1)$ is independent of $X$. Find explicit expressions for $E[X|Y]$ and $E[X^2|Y]$.

2 Estimation of a signal from multiple noisy measurements
Let $Y_i = X + Z_i$, $i = 1, \ldots, n$, where $X$ and the $Z_i$'s are mutually independent, zero-mean random variables with finite second moments. The variance of $X$ is $\sigma_X^2$, and the variance of each $Z_i$ is $\sigma_Z^2$.

(a) Find the linear MMSE estimator $\hat{E}[X|Y_1, \ldots, Y_n]$ of $X$ given the observations $Y_1, \ldots, Y_n$.

(b) Find the resulting MSE $E\left[\left( X - \hat{E}[X|Y_1, \ldots, Y_n] \right)^2 \right]$.

3 Data processing and MMSE estimation
Let $X$, $Y$ and $Z$ be three random variables with finite second moments, defined on a common probability space $(\Omega, \mathcal{F}, P)$. We say that $X$ and $Z$ are conditionally independent given $Y$ (and write $X \rightarrow Y \rightarrow Z$) if for any two events $A, B \in \mathcal{F}$,

$$P(X \in A, Z \in B|Y) = P(X \in A|Y)P(Z \in B|Y)$$

where $P(X \in A, Z \in B|Y) = E[1_{A \times B}|Y]$. If $X$, $Y$, and $Z$ have a joint pdf $f_{X,Y,Z}$, then conditional independence is equivalent to

$$f_{X|Z|Y}(x, z|y) = f_{X|Y}(x|y)f_{Z|Y}(z|y).$$

When working on this problem, you may assume that all necessary joint pdfs exist.

(a) Given $X$, let $Y$ and $Z$ be defined by $Y = X + W_1$ and $Z = Y + W_2$, where $X$, $W_1$, and $W_2$ are mutually independent random variables. Prove that $X \rightarrow Y \rightarrow Z$.

(b) Show that $X \rightarrow Y \rightarrow Z$ implies that $E[X|Y, Z] = E[X|Y]$.

(c) Show that if $X \rightarrow Y \rightarrow Z$, then

$$\text{MMSE}(X|Z) - \text{MMSE}(X|Y) = E \left[ (E[X|Y] - E[X|Z])^2 \right],$$


where, for any two second-order random variables $U$ and $V$, we have defined

$$\text{MMSE}(U|V) \triangleq \mathbb{E}\left[(U - \mathbb{E}[U|V])^2\right].$$

(d) Use your answer from part (c) to deduce the *data processing inequality for MMSE estimation*:

if $X \rightarrow Y \rightarrow Z$, then

$$\text{MMSE}(X|Y) \leq \text{MMSE}(X|Z). \quad (1)$$

What is the necessary and sufficient condition for equality to be achieved in (1)?

(e) Let $X$ be a random variable with finite second moment, and let $Z \sim N(0,1)$ be independent of $X$. For each $\rho > 0$, let $Y_\rho = X + \sqrt{\rho}Z$. Use the data processing inequality from part (d) to show that the function

$$E(\rho) \triangleq \text{MMSE}(X|Y_\rho)$$

is increasing.