ECE 534 RANDOM PROCESSES **FALL 2012**
PROBLEM SET 3 **Due beginning of class, Tuesday, October 30**

3. MMSE Estimation

Assigned Reading: Chapter 3 of the course notes.

Reminders: The first midterm exam will be on Wednesday, October 24, from 6 pm until 7:30 pm in 165 Everitt Lab.

Problems to be handed in:

Problems 3.1 and 3.5 from the course notes, as well as the following three problems:

1 Estimation of a symmetric binary signal in Gaussian noise

Let X take values in the set $\{-1, +1\}$, where each of the two possible values has probability $1/2$. Consider the problem of estimating X on the basis of an observation $Y = X + \sqrt{\rho}Z$, where $\rho > 0$ is a deterministic parameter that quantifies the signal-to-noise ratio, and the additive noise $Z \sim N(0, 1)$ is independent of X. Find explicit expressions for $\mathbb{E}[X|Y]$ and $\widehat{\mathbb{E}}[X|Y]$.

2 Estimation of a signal from multiple noisy measurements

Let $Y_i = X + Z_i$, $i = 1, \ldots, n$, where X and the Z_i's are mutually independent, zero-mean random variables with finite second moments. The variance of X is σ_X^2, and the variance of each Z_i is σ_Z^2.

(a) Find the linear MMSE estimator $\widehat{\mathbb{E}}[X|Y_1, \ldots, Y_n]$ of X given the observations $Y_1, \ldots, Y_n$.

(b) Find the resulting MSE $\mathbb{E}\left[\left(X - \widehat{\mathbb{E}}[X|Y_1, \ldots, Y_n]\right)^2\right]$.

3 Data processing and MMSE estimation

Let X, Y and Z be three random variables with finite second moments, defined on a common probability space $(\Omega, \mathcal{F}, P)$. We say that X and Z are *conditionally independent given* Y (and write $X \to Y \to Z$) if for any two events $A, B \in \mathcal{F}$,

$$P(X \in A, Z \in B|Y) = P(X \in A|Y)P(Z \in B|Y)$$

where $P(X \in A, Z \in B|Y) = \mathbb{E}[1_{A\times B}|Y]$. If X, Y, and Z have a joint pdf f_{XYZ}, then conditional independence is equivalent to

$$f_{XZ|Y}(x, z|y) = f_{X|Y}(x|y)f_{Z|Y}(z|y).$$

When working on this problem, you may assume that all necessary joint pdf's exist.

(a) Given X, let Y and Z be defined by $Y = X + W_1$ and $Z = Y + W_2$, where X, W_1, and W_2 are mutually independent random variables. Prove that $X \to Y \to Z$.

(b) Show that $X \to Y \to Z$ implies that $\mathbb{E}[X|Y, Z] = \mathbb{E}[X|Y]$.

(c) Show that if $X \to Y \to Z$, then

$$\text{MMSE}(X|Z) - \text{MMSE}(X|Y) = \mathbb{E}\left[(\mathbb{E}[X|Y] - \mathbb{E}[X|Z])^2\right],$$

where, for any two second-order random variables U and V, we have defined

$$\mathrm{MMSE}(U|V) \triangleq \mathbb{E}\left[(U - \mathbb{E}[U|V])^2\right].$$

(d) Use your answer from part (c) to deduce the *data processing inequality for MMSE estimation*: if $X \to Y \to Z$, then

$$\mathrm{MMSE}(X|Y) \le \mathrm{MMSE}(X|Z). \tag{1}$$

What is the necessary and sufficient condition for equality to be achieved in (1)?

(e) Let X be a random variable with finite second moment, and let $Z \sim N(0,1)$ be independent of X. For each $\rho > 0$, let $Y_\rho = X + \sqrt{\rho}Z$. Use the data processing inequality from part (d) to show that the function

$$E(\rho) \triangleq \mathrm{MMSE}(X|Y_\rho)$$

is increasing.