ECE 534 RANDOM PROCESSES **FALL 2012**
PROBLEM SET 2 **Due beginning of class, Thursday, October 11**

2. Convergence of a Sequence of Random Variables

Assigned Reading: Chapter 2 of the course notes and supplemental material to be posted on the course webpage.

Reminders: No class on Tuesday, October 2, and Thursday, October 4, due to the Allerton Conference. We will have a make-up class on Monday, October 1, from 3 pm until 4:50 pm in 165 Everitt Lab.

Problems to be handed in:

1 Stochastic convergence of sample variance

Let $X_1, X_2, \ldots$ are i.i.d. random variables with finite mean $\mu = \mathbb{E}[X_1]$ and finite variance $\sigma^2 = \text{Var}[X_1]$. Let

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

be the sample mean, and let

$$\overline{V}_n = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X}_n)^2$$

be the sample variance.

(a) Show that $\mathbb{E}\left[\overline{V}_n\right] = \sigma^2$.

(b) Show that $\overline{V}_n \xrightarrow{\text{p.}} \sigma^2$.

2 Wild oscillations

Let $X_1, X_2, \ldots$ be a sequence of random variables, such that

$$X_n = \begin{cases} \frac{1}{n}, & \text{with probability } 1 - \frac{1}{n^2} \\ n, & \text{with probability } \frac{1}{n^2} \end{cases}$$

In which of the four senses (a.s., m.s., p., d.) does X_n converge? Justify your answers.

3 Convergence to a constant in mean square sense

Let $X_1, X_2, \ldots$ be a sequence of random variables, and let $c \in \mathbb{R}$ be a constant. Prove that $X_n \xrightarrow{\text{m.s.}} c$ if and only if

$$\lim_{n\to\infty} \mathbb{E}[X_n] = c \qquad \text{and} \qquad \lim_{n\to\infty} \text{Var}[X_n] = 0.$$

4 Stochastic convergence of a minimum

Let $Z_1, Z_2, \ldots$ be i.i.d. random variables with pdf f. Suppose that $P(Z_1 > 0) = 1$, and that $\lambda = \lim_{x\to 0^+} f(x) > 0$. Consider the sequence of random variables $\{X_n\}_{n=1}^{\infty}$ defined by

$$X_n = n \min\{Z_1, \ldots, Z_n\}.$$

Prove that $X_n \xrightarrow{\text{d.}} Z$, where Z has $\text{Exp}(\lambda)$ distribution. *Hint:* Lemma 2.3.1 in the lecture notes may be useful.

5 Subgaussian random variables

A random variable X is called *subgaussian* if there exists a constant $c > 0$ such that

$$\mathbb{E}\left[e^{t(X-\mathbb{E}[X])}\right] \leq e^{ct^2/2}, \qquad \forall t \in \mathbb{R}.$$

(the constant $\sqrt{c}$ is often called the *scale factor of* X.)

(a) Are the following random variables subgaussian? Why or why not?

(i) $X \sim N(\mu, \sigma^2)$ for some $\mu \in \mathbb{R}$ and $\sigma^2 > 0$

(ii) $X \in \{-M, M\}$ for some $M > 0$, taking the values $-M$ and M with equal probability

(iii) $X \sim \text{Exp}(\lambda)$ for some $\lambda > 0$

(b) Prove that if $X_1, \ldots, X_n$ are independent subgaussian random variables with the same scale factor c, then $Y = a_1X_1 + \ldots + a_nX_n$ is subgaussian with the same scale factor for any $a_1, \ldots, a_n \in \mathbb{R}$ satisfying $a_1^2 + \ldots + a_n^2 = 1$.

(c) Let $X_1, X_2, \ldots$ be a sequence of i.i.d. subgaussian random variables. Prove that the sequence of the sample means $\overline{X}_n = n^{-1}\sum_{i=1}^n X_i$ converges to $\mu = \mathbb{E}[X_1]$ almost surely.