ECE 534 RANDOM PROCESSES **FALL 2012**
PROBLEM SET 1 **Due beginning of class, Thu Sep 13**

Please visit the course website soon: http://maxim.ece.illinois.edu/fall12/

1. Review of Basic Probability

Assigned reading: Chapter 1 of the ECE 534 course notes, including the problems at the end of the chapter. A complete version of the prerequisite probability material you will be expected to know for this course is given in the course notes for ECE 313, Chapter 1, Sections 2.1-7, 2.9, 2.10, 3.1-6, 3.8, 4.1-8, and wrap-up in Chapter 5, except estimation and reliability concepts. Also, the past ECE 313 homeworks, exams, and their solutions are available on the ECE 313 website.

Problems to be handed in:

1 Persistence and coin tossing
A persistent individual keeps tossing a fair coin until it comes up heads twice in a row.

(a) Describe the sample space Ω.

(b) What is the probability that only k tosses are required?

2 Independence and extreme events
Prove that any event E that has either $P(E) = 0$ or $P(E) = 1$ is independent of all other events.

3 Some conditional probabilities
Consider a probability space $(\Omega, \mathcal{F}, P)$, and let A, B, C be some events such that $P(A \cap B|C) = 1$. Label each of the following statements as either true or false and justify your answers:

(a) $P(A \cap B) = 1$

(b) $P(A \cap B \cap C) = P(C)$

(c) $P(A^c|C) = 0$

(d) $C = \Omega$

4 Some more conditional probabilities
Let $(\Omega, \mathcal{F}, P)$ be a probability space. Consider an event $B \in \mathcal{F}$ with $P(B) > 0$. Define a function $P_B : \mathcal{F} \to [0, 1]$ that takes each $A \in \mathcal{F}$ to $P_B(A) := P(A|B)$. Prove that P_B is itself a probability measure on $(\Omega, \mathcal{F})$, i.e., that it satisfies the axioms **P.1**–**P.3**.

5 Conditional independence
Consider three events A, B, C with nonzero probabilities, and suppose that A and C are *conditionally independent given* B, i.e., $P(A \cap C|B) = P(A|B)P(C|B)$.

(a) Prove that $P(A \cap B \cap C) = P(A)P(B|A)P(C|B) = P(C)P(B|C)P(A|B)$.

(b) Prove that $P(A|B, C) = P(A|B)$.

(c) True or false: if A and C are conditionally independent given B, then A and B are also conditionally independent given C. Justify your answer.

6 Partitions
Suppose that k events $E_1, \dots, E_k$ form a partition of the sample space Ω. Let A be an event with $P(A) > 0$. Prove that if $P(E_1|A) < P(E_1)$, then there must be some $i \in \{2, \dots, k\}$ such that $P(E_i|A) > P(E_i)$.

7 Nonlinear transformation of a random variable

Let X be a $U(-1, 1)$ random variable. Define a new random variable Y by

$$Y = \begin{cases} X, & \text{if } X \leq 0 \\ 1, & \text{if } X > 0 \end{cases}$$

(a) Find the cdf F_Y and sketch its graph.

(b) Find the pdf f_Y.

(c) Find the mean and the variance of Y

(d) Find the conditional expectation $\mathbb{E}[X|Y = y]$ for all values of y, for which it is defined.

8 Functions of a Gaussian random variable

Let X be a Gaussian random variable with zero mean and variance σ^2.

(a) Find $\mathbb{E}[\cos(nX)]$ for $n = 1, 2, \ldots$

(b) Find $\mathbb{E}[X^n]$ for $n = 1, 2, \ldots$

9 Two friends on a hot summer day

Bart and Milhouse are bored, and they decide to play a game. They start at 742 Evergreen Terrace. Milhouse flips a fair coin. If the coin comes up heads, then Bart generates a sample X of an Exp(1) random variable, hops on his skateboard, and then skates the distance of X feet down the street. If the coin comes up tails, then Bart stays where he is. Let Y be the distance Bart has skated.

(a) Find the cdf F_Y and sketch its graph.

(b) Find the pdf f_Y (Hint: Dirac delta functions may be involved.)

(c) Find the mean and the variance of Y.

(d) Principal Skinner is standing on Evergreen Terrace just 3 feet away from where Bart and Milhouse are, staring absentmindedly at his shoes. What is the probability that Bart will not collide with him? (Assume that Bart cannot get past the Principal without a collision.)

10 The magic of Poissonization

Suppose we have a coin with probability p of coming up heads.

(a) We toss the coin n times. Let X denote the number of heads, and let Y be the number of tails. Prove that X and Y are dependent random variables.

(b) Now let N be a Poisson(λ) random variable. We toss the coin N times. Again, let X be the number of heads, and let Y be the number of tails. Prove that X and Y are independent random variables.