University of Illinois Fall 2012

ECE 534: Quiz

Wednesday September 19, 2012
6:00 p.m. — 7:30 p.m.
116 Roger Adams Laboratory

1. **[10 points]** Alice, Bob and Eve have two standard decks of playing cards. Every day, they play the following game. Alice takes the first deck, pulls out a card at random, and shows it to Eve. Eve announces the suit of the card to everyone. Bob removes all cards of that suit from the second deck, gives the remaining cards a nice thorough shuffle, and then hands them to Alice. Finally, Alice draws a random card from this second, smaller deck. After each game, the decks are reassembled and reshuffled.

   (a) (2 pts.) On Monday, Eve announces that the suit of the card Alice has pulled from the first deck is clubs. What is the conditional probability that Alice has drawn an ace?

   Solution: Denote by C_1 the event that Eve has announced "clubs," and by A_1 the event that the card Alice has drawn from the first deck is an ace. Then A_1 and C_1 are independent because knowing the suit tells you nothing about which card has been drawn, and vice versa. Consequently, $P(A_1|C_1) = P(A_1) = 4/52 = 1/13$ (there are four aces in the deck).

   If you don't recognize the independence, then you can get the same answer by using the definition:

$$P(A_1|C_1) = \frac{P(A_1 \cap C_1)}{P(C_1)} = \frac{1/52}{13/52} = \frac{1}{13}.$$

   (b) (8 pts.) A new game is played on Tuesday. What is the probability that the suit of the card Alice has pulled from the second, smaller deck is clubs?

   Solution: The key piece of information here is that this is a new game, independent from the one on Monday. Consequently, we *do not know* which suit Eve has announced, so we have to account for the possibility that Eve may have announced some other suit instead of "clubs."

   With this in mind, let C_1 be the event that Eve has announced clubs, and let C_2 be the event that Alice has pulled clubs from the second, smaller deck. Then $P(C_1) = 1/4$, $P(C_2|C_1) = 0$ and $P(C_2|C_1^c) = 13/39 = 1/3$. By the law of total probability,

$$P(C_2) = P(C_2|C_1)P(C_1) + P(C_2|C_1^c)P(C_1^c) = 0 \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}.$$

2. **[10 points]** Let X_1 and X_2 be two independent random variables taking values in $\{0, 1\}$, where $P(X_i = 0) = P(X_i = 1) = 1/2$ for $i = 1, 2$. Let $X_3 = X_1 \oplus X_2$, i.e., the Boolean exclusive-OR (XOR) of X_1 and X_2.

   (a) (5 pts.) Are X_1, X_2 and X_3 mutually independent? Justify your answer.

   Solution: They are not because X_3 is a deterministic function of X_1 and X_2.

   (b) (5 pts.) Are X_1, X_2 and X_3 pairwise independent? Justify your answer.

   Solution: They are: X_1 and X_2 are independent by construction. For X_1 and X_3 we have, for any $a, b \in \{0,1\}$

$$\begin{aligned}
P(X_1 = a, X_3 = b) &= P(X_1 = a, X_1 \oplus X_2 = b) \\
&= P(X_1 = a, X_2 \oplus a = b) \\
&= P(X_1 = a, X_2 = a \oplus b) \\
&= P(X_1 = a)P(X_2 = a \oplus b) \\
&= \frac{1}{2} \cdot \frac{1}{2},
\end{aligned}$$

where the last two steps use the fact that X_1 and X_2 are independent, and both are Bernoulli(1/2) random variables. We also have, for any $a \in \{0,1\}$

$$\begin{aligned}
&P(X_3 = a) \\
&= P(X_1 \oplus X_2 = a) \\
&= P(X_1 \oplus X_2 = a, X_1 = X_2) + P(X_1 \oplus X_2 = a, X_1 \neq X_2) \\
&= P(X_1 = X_2)P(X_1 \oplus X_2 = a | X_1 = X_2) + P(X_1 \neq X_2)P(X_1 \oplus X_2 = a | X_1 \neq X_2) \\
&= \frac{1}{2} \cdot 1_{\{a=0\}} + \frac{1}{2} \cdot 1_{\{a=1\}} \\
&= \frac{1}{2}.
\end{aligned}$$

Thus, X_3 is also a Bernoulli(1/2) random variable, so we can write

$$P(X_1 = a, X_3 = b) = \frac{1}{2} \cdot \frac{1}{2} = P(X_1 = a)P(X_3 = b)$$

for all $a, b \in \{0,1\}$, so X_1 and X_3 are independent. By symmetry, so are X_2 and X_3. Thus, X_1, X_2, X_3 are pairwise independent.

3. **[10 points]** Given a random variable X, two random variables W and V (considered to be noise) are added to X to form the observations

$$\begin{aligned}
Y &= X + V \\
Z &= X + W.
\end{aligned}$$

Both W and V are assumed to be independent of X. Suppose that all variables have mean 0, that X has variance σ_X^2, and that the random variables W and V have equal variance σ^2. Finally, suppose that W and V are such that

$$\mathbb{E}[VW] = \rho\sigma^2,$$

where $\rho \in [-1, 1]$ is called the *correlation coefficient*.

(a) (3 pts.) Find $\mathbb{E}[YZ]$.
Solution:

$$\begin{aligned}
\mathbb{E}[YZ] &= \mathbb{E}[(X+V)(X+W)] \\
&= \mathbb{E}[X^2 + XV + XW + VW] \\
&= \mathbb{E}[X^2] + \mathbb{E}[XV] + \mathbb{E}[XW] + \mathbb{E}[VW] && \text{(linearity of expectation)} \\
&= \sigma_X^2 + 0 + 0 + \rho\sigma^2 && \text{(X is independent of V, W; all have mean 0)}
\end{aligned}$$

Thus, $\mathbb{E}[YZ] = \sigma_X^2 + \rho\sigma^2$.

(b) (7 pts.) Suppose that we cannot observe X, but wish to estimate it based on the observations Y and/or Z. Two possible estimates are

$$\widehat{X}_1 = Y \qquad \text{and} \qquad \widehat{X}_2 = \frac{Y+Z}{2}$$

In other words, the estimate $\widehat{X}_1$ only makes use of one observation, while $\widehat{X}_2$ looks at both. For what values of ρ is $\widehat{X}_2$ the better estimate of the two in the sense that

$$\mathbb{E}\left[(X-\widehat{X}_2)^2\right] < \mathbb{E}\left[(X-\widehat{X}_1)^2\right]?$$

Solution: We have

$$\mathbb{E}\left[(X-\widehat{X}_1)^2\right] = \mathbb{E}[(X-Y)^2] = \mathbb{E}[V^2] = \sigma^2$$

and

$$\begin{aligned}
\mathbb{E}\left[(X-\widehat{X}_2)^2\right] &= \mathbb{E}\left[\left(X - \frac{Y+Z}{2}\right)^2\right] \\
&= \mathbb{E}\left[\left(\frac{V+W}{2}\right)^2\right] \\
&= \frac{1}{4}\mathbb{E}[V^2 + 2VW + W^2] \\
&= \frac{1}{4}\left(\sigma^2 + 2\rho\sigma^2 + \sigma^2\right) \\
&= \frac{(1+\rho)\sigma^2}{2}
\end{aligned}$$

Since $\rho \in [-1,1]$, $\widehat{X_2}$ will be strictly better than $\widehat{X}_1$ if and only if $\rho \in [-1,1)$.

4. **[10 points]** Let X and Y be the coordinates of a point selected uniformly at random from the triangle with vertices $(0,0)$, $(0,1)$ and $(1,0)$ in $\mathbb{R}^2$.

(a) (4 pts.) Find the pdf of X.

Solution: Let $T \subset \mathbb{R}^2$ denote the set of all points on the triangle with vertices $(0,0)$, $(0,1)$, and $(1,0)$. Then

$$T = \left\{(x,y) \in \mathbb{R}^2 : 0 \le x \le 1,\ 0 \le y \le 1-x\right\},$$

and the area of T is $1/2$. The joint pdf of X and Y is

$$f_{XY}(x,y) = \begin{cases} 2, & \text{if } (x,y) \in T \\ 0, & \text{otherwise} \end{cases}$$

If $x < 0$ or $x > 1$, $f_{XY}(x,y) = 0$, so $f_X(x) = 0$ as well. If $0 \le x \le 1$, then we must integrate f_{XY} with respect to y:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,\mathrm{d}y = \int_{\{y:(x,y)\in T\}} 2\,\mathrm{d}y = 2\int_0^{1-x} \mathrm{d}y = 2(1-x).$$

Thus,

$$f_X(x) = 2(1-x)I_{[0,1]}(x).$$

(b) (2 pts.) Are X and Y independent? Briefly justify your answer.

Solution: No. By symmetry, $f_Y(y) = 2(1-y)I_{[0,1]}(y)$, and the joint pdf f_{XY} is not equal to the product of the marginal pdfs f_X and f_Y.

(c) (4 pts.) Find $\mathbb{E}[X^2|Y = y]$ for $y \in (0, 1)$.

Solution: By inspection, the conditional distribution of X given $Y = y$ for $0 < y < 1$ is uniform on the interval $[0, 1-y]$. In other words,

$$f_{X|Y}(x|y) = \frac{1}{1-y} I_{[0,1-y]}(x).$$

Therefore,

$$\mathbb{E}[X^2|Y = y] = \int_{-\infty}^{\infty} x^2 f_{X|Y}(x|y)\,\mathrm{d}x = \frac{1}{1-y}\int_0^{1-y} x^2\,\mathrm{d}x = \frac{(1-y)^2}{3}.$$