University of Illinois Fall 2012

ECE 534: Quiz

Wednesday September 19, 2012
6:00 p.m. — 7:30 p.m.
116 Roger Adams Laboratory

Name: _____

University NetID: _____

- You have one hour and 30 minutes for this quiz. The quiz is closed book and closed note.
- Calculators, laptop computers, smartphones, PDAs, two-way e-mail pagers, etc. may not be used.
- Write your answers in the spaces provided.
- **Please show all of your work. Answers without appropriate justification will receive very little credit.** If you need extra space, use the back of the previous page.

Scores	
1. 10 points	
2. 10 points	
3. 10 points	
4. 10 points	
Total (40 points)	

1. **[10 points]** Alice, Bob and Eve have two standard decks of playing cards. Every day, they play the following game. Alice takes the first deck, pulls out a card at random, and shows it to Eve. Eve announces the suit of the card to everyone. Bob removes all cards of that suit from the second deck, gives the remaining cards a nice thorough shuffle, and then hands them to Alice. Finally, Alice draws a random card from this second, smaller deck. After each game, the decks are reassembled and reshuffled.

   (a) (2 pts.) On Monday, Eve announces that the suit of the card Alice has pulled from the first deck is clubs. What is the conditional probability that Alice has drawn an ace?

   (b) (8 pts.) A new game is played on Tuesday. What is the probability that the suit of the card Alice has pulled from the second, smaller deck is clubs?

2. **[10 points]** Let X_1 and X_2 be two independent random variables taking values in $\{0, 1\}$, where $P(X_i = 0) = P(X_i = 1) = 1/2$ for $i = 1, 2$. Let $X_3 = X_1 \oplus X_2$, i.e., the Boolean exclusive-OR (XOR) of X_1 and X_2.

   (a) (5 pts.) Are X_1, X_2 and X_3 mutually independent? Justify your answer.

   (b) (5 pts.) Are X_1, X_2 and X_3 pairwise independent? Justify your answer.

3. **[10 points]** Given a random variable X, two random variables W and V (considered to be noise) are added to X to form the observations

$$Y = X + V$$
$$Z = X + W.$$

Both W and V are assumed to be independent of X. Suppose that all variables have mean 0, that X has variance σ_X^2, and that the random variables W and V have equal variance σ^2. Finally, suppose that W and V are such that

$$\mathbb{E}[VW] = \rho\sigma^2,$$

where $\rho \in [-1, 1]$ is called the *correlation coefficient*.

(a) (3 pts.) Find $\mathbb{E}[YZ]$.

(b) (7 pts.) Suppose that we cannot observe X, but wish to estimate it based on the observations Y and/or Z. Two possible estimates are

$$\widehat{X}_1 = Y \qquad \text{and} \qquad \widehat{X}_2 = \frac{Y+Z}{2}$$

In other words, the estimate $\widehat{X}_1$ only makes use of one observation, while $\widehat{X}_2$ looks at both. For what values of ρ is $\widehat{X}_2$ the better estimate of the two in the sense that

$$\mathbb{E}\left[(X - \widehat{X}_2)^2\right] < \mathbb{E}\left[(X - \widehat{X}_1)^2\right]?$$

4. **[10 points]** Let X and Y be the coordinates of a point selected uniformly at random from the triangle with vertices $(0,0)$, $(0,1)$ and $(1,0)$ in $\mathbb{R}^2$.

   (a) (4 pts.) Find the pdf of X.

   (b) (2 pts.) Are X and Y independent? Briefly justify your answer.

   (c) (4 pts.) Find $\mathbb{E}[X^2|Y=y]$ for $y \in (0,1)$.