University of Illinois Fall 2012

ECE 534: Exam II

Wednesday November 28, 2012
6:00 p.m. — 7:30 p.m.
165 Everitt Laboratory

1. **[25 points]** Consider the following continuous-time Markov process $Y = (Y_t)_{t \ge 0}$ where each Y_t takes values in $\{0, 1\}$. The distribution of Y_0 is fixed. Let $N = (N_t)_{t \ge 0}$ be a Poisson process with unit rate, independent of Y_0. The value of Y_t remains constant between successive count times of N. With each new count, it flips with probability 1/2 and stays the same with probability 1/2.

   (a) Show that Y is time-homogeneous, and find its transition matrix $H(t)$ and generator matrix Q.
   Solution: The probability of state transition from $Y_s = i$ to $Y_t = j$ for any $0 \le s < t$ and any $i, j \in \{0, 1\}$ depends only on the number of counts $N_t - N_s$, which is a Poisson random variable with parameter 1. Hence, Y is time-homogeneous.
   To find the transition matrix $H(t)$, it is convenient to condition on N_t. Let $H(t|n)$ denote the matrix of conditional probabilities of state transitions given $N_t = n$. Then

$$
\begin{aligned}
H(t|0) &= I \\
H(t|n) &= \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}^n = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}, \qquad n = 1, 2, \ldots
\end{aligned}
$$

   Therefore,

$$
\begin{aligned}
H(t) &= \sum_{n=0}^{\infty} H(t|n) P(N_t = n) \\
&= e^{-t} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (1 - e^{-t}) \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1+e^{-t}}{2} & \frac{1-e^{-t}}{2} \\ \frac{1-e^{-t}}{2} & \frac{1+e^{-t}}{2} \end{pmatrix} \\
Q(t) &= \frac{\mathrm{d}}{\mathrm{d}t} H(t)\Big|_{t=0} = \begin{pmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}
\end{aligned}
$$

   (b) Find the invariant distribution of Y (if any).
   Solution: For a two-state continuous-time Markov process with generator matrix $Q = \begin{pmatrix} -\alpha & \alpha \\ \beta & -\beta \end{pmatrix}$, the invariant distribution is $\pi = \left(\frac{\beta}{\alpha+\beta} \; \frac{\alpha}{\alpha+\beta} \right)$. In this case, $\pi = (1/2 \; 1/2)$.

2. **[25 points]** Let $\mathcal{X}$ and $\mathcal{S}$ be two finite sets, and consider two discrete-time random processes, $X = (X_k)_{k \in \mathbb{Z}_+}$ taking values in $\mathcal{X}$ and $S = (S_k)_{k \in \mathbb{Z}_+}$ taking values in $\mathcal{S}$. The processes are specified as follows. We have a probability distribution μ on $\mathcal{S}$, a collection $(\pi^{(s)} : s \in \mathcal{S})$ of probability distributions on $\mathcal{X}$, and an update function $f : \mathcal{S} \times \mathcal{X} \to \mathcal{S}$. At time $k = 0$, S_0 has distribution μ and X_0 has distribution $\pi^{(S_0)}$. At each subsequent time $k = 1, 2, \ldots$,

$$
S_k = f(S_{k-1}, X_{k-1}) \qquad \text{and} \qquad X_k \sim \pi^{(S_k)}.
$$

(a) For each $k \geq 0$, let $Y_k = \binom{X_k}{S_k}$. Show that the process $Y = (Y_k)_{k\in\mathbb{Z}_+}$ taking values in $\mathcal{X} \times \mathcal{S}$ and the process $S = (S_k)_{k\in\mathbb{Z}_+}$ are Markov.

Solution: Since the distribution of X_k for each k is determined by S_k, there exists a function $\Phi : \mathcal{S} \times [0, 1] \to \mathcal{X}$, such that $X_k = \Phi(S_k, U_k)$, where $U = (U_k)_{k\in\mathbb{Z}_+}$ is a sequence of i.i.d. $U(0, 1)$ random variables such that U_k is independent of $S_1, \ldots, S_k$. Therefore,

$$Y_k = \begin{pmatrix} X_k \\ S_k \end{pmatrix} = \begin{pmatrix} \Phi(S_k, U_k) \\ f(S_{k-1}, X_{k-1}) \end{pmatrix} = \begin{pmatrix} \Phi(f(S_{k-1}, X_{k-1}), U_k) \\ f(S_{k-1}, X_{k-1}) \end{pmatrix} \equiv \Psi(Y_{k-1}, U_k)$$

for a suitable function $\Psi : \mathcal{X} \times \mathcal{S} \times [0, 1] \to \mathcal{X} \times \mathcal{S}$. Therefore, Y is a Markov process. Similarly, $S_k = f(S_{k-1}, X_{k-1}) = f(S_{k-1}, \Phi(S_{k-1}, U_k)) \equiv \Lambda(S_{k-1}, U_k)$ for a suitable function $\Lambda : \mathcal{S} \times [0, 1] \to \mathcal{S}$. Therefore, S is Markov.

(b) Is the process $X = (X_k)_{k\in\mathbb{Z}_+}$ necessarily Markov? Justify your answer. If the answer is 'no,' give a sufficient condition for X to be Markov. (Hint: think about the update function f.)

Solution: No, the process X is not necessarily Markov. Indeed, at each k we have

$$\begin{aligned} X_k &= \Phi(S_k, U_k) \\ &= \Phi(f(S_{k-1}, X_{k-1}), U_k) \\ &= \Phi(f(S_{k-1}, \Phi(X_{k-2}, U_{k-1})), U_k) \\ &= \Phi(f(f(S_{k-2}, X_{k-2}), \Phi(X_{k-2}, U_{k-1}))), U_k) \\ &= \ldots \end{aligned}$$

so X_k may depend on X_{k-2}. In fact, X may have arbitrarily long memory. One way to guarantee that X is Markov is to let the update function $f(s, x)$ depend only on s or only on x. In the first case,

$$X_k = \Phi(S_k, U_k) = \Phi(f(S_{k-1}), U_k)$$

independently of X^{k-1}. In the second case,

$$X_k = \Phi(S_k, U_k) = \Phi(f(X_{k-1}), U_k),$$

independently of X^{k-2}.

3. **[25 points]** Let $X = (X_k)_{k\in\mathbb{Z}_+}$ and $Y = (Y_k)_{k\in\mathbb{Z}_+}$ be two discrete-time second-order random processes on a common probability space. We say that Y is a *martingale with respect to* X if the following two conditions hold:

- for each k, Y_k is a function of $X^k = (X_1, \ldots, X_k)$
- for each k, $\mathbb{E}[Y_{k+1}|X^k] = Y_k$

(a) Let $W, X_1, X_2, \ldots$ be second-order random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$. For each k, let $Y_k = \mathbb{E}[W|X^k]$. Prove that $Y = (Y_k)_{k\in\mathbb{Z}_+}$ is a martingale with respect to $X = (X_k)_{k\in\mathbb{Z}_+}$.

Solution: The conditional expectation $\mathbb{E}[W|X^k]$ is, by definition, a function of X^k. Moreover, by the tower property of conditional expectation,

$$\mathbb{E}[Y_{k+1}|X^k] = \mathbb{E}[\mathbb{E}[W|X^{k+1}]|X^k] = \mathbb{E}[\mathbb{E}[W|X^k, X_{k+1}]|X^k] = \mathbb{E}[W|X^k] = Y_k.$$

Thus, Y is a martingale with respect to X.

(b) Suppose that the random variables $W, X_1, X_2, \dots$ from part (a) are jointly Gaussian and zero-mean, and that the X_k's are orthogonal. Show that the process Y from part (a) has the recursive representation

$$Y_{k+1} = Y_k + \mathbb{E}[W|X_{k+1}], \qquad k = 0, 1, 2, \dots$$

Solution: Since all relevant random variables are jointly Gaussian and zero mean, we have $Y_k = \mathbb{E}[W|X^k] = \widehat{\mathbb{E}}[W|X^k]$ and $\mathbb{E}[W|X_k] = \widehat{\mathbb{E}}[W|X_k]$. Moreover, because the X_k's are orthogonal,

$$Y_{k+1} = \widehat{\mathbb{E}}[W|X^{k+1}] = \sum_{i=1}^{k+1} \widehat{\mathbb{E}}[W|X_i] = \sum_{i=1}^{k} \widehat{\mathbb{E}}[W|X_i] + \widehat{\mathbb{E}}[W|X_{k+1}] = Y_k + \mathbb{E}[W|X_{k+1}]$$

4. **[25 points]** Let $X_1, X_2, \dots$ be a sequence of i.i.d. $N(1,1)$ random variables, and let $N = (N_t)_{t\geq 0}$ be a Poisson process with rate 2, where we assume that N is independent of $X_1, X_2, \dots$. Define the continuous-time compound process $Y = (Y_t)_{t\geq 0}$ by

$$Y_t = \begin{cases} 0, & t = 0 \\ \sum_{i=1}^{N_t} X_i, & t > 0 \end{cases}$$

(a) Find the mean function of $Y = (Y_t)_{t\geq 0}$.
Solution:

$$\begin{aligned}
\mathbb{E}[Y_t] &= \mathbb{E}[\mathbb{E}[Y_t|N_t]] && \text{(tower property)} \\
&= \mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^{N_t} X_i \middle| N_t\right]\right] && \text{(definition of } Y_t\text{)} \\
&= \mathbb{E}\left[\sum_{i=1}^{N_t} \mathbb{E}X_i\right] && \text{(independence of } N \text{ and } X\text{)} \\
&= \mathbb{E}N_t \cdot \mathbb{E}X_1 && (X_i\text{'s are i.i.d.)} \\
&= 2t \cdot 1 && (N_t \text{ is Poisson}(2t) \text{ and } X_1 \text{ is } N(1,1)) \\
&= 2t.
\end{aligned}$$

(b) Show that Y has independent increments.
Solution: For each n and each $0 = t_0 = t_1 < t_2 < \dots < t_n$, let $I_k = Y_{t_k} - Y_{t_{k-1}}$ for $k = 1, \dots, n$. Then

$$I_k = \sum_{i=1}^{N_{t_k}} X_i - \sum_{i=1}^{N_{t_{k-1}}} X_i = \sum_{i=N_{t_{k-1}}+1}^{N_{t_k}} X_i.$$

Thus, each I_k depends on $N_{t_{k-1}}$, on the increment $N_{t_k} - N_{t_{k-1}}$, and on $X_{N_{t_{k-1}}+1}, \dots, X_{N_{t_k}}$. Thus, $I_1, \dots, I_n$ are mutually conditionally independent given $N_{t_0} = N_0 = 0$ and the increments $N_{t_k} - N_{t_{k-1}}$, $k = 1, \dots, n$. In fact, I_k is conditionally independent of all other I_j's given $N_{t_{k-1}}$ and $N_{t_k} - N_{t_{k-1}}$. Now, since N has independent increments, I_k is conditionally independent of all other I_j's given $N_{t_0} = N_0 = 0$, $N_{t_1} - N_{t_0}$, $N_{t_2} - N_{t_1}, \dots, N_{t_k} - N_{t_{k-1}}$. Since these increments of N are independent, $I_1, \dots, I_n$ are independent as well.