University of Illinois Fall 2012

ECE 534: Exam II

Wednesday November 28, 2012
6:00 p.m. — 7:30 p.m.
165 Everitt Laboratory

Name: ____

University NetID: __________________________

- You have ninety minutes for this exam. The exam is closed book and closed note, however, you may consult both sides of two $8.5'' \times 11''$ sheets of notes.
- Calculators, laptop computers, smartphones, PDAs, two-way e-mail pagers, etc. may not be used.
- Write your answers in the spaces provided.
- **Please show all of your work. Answers without appropriate justification will receive very little credit.** If you need extra space, use the back of the previous page.

Scores

1. 25 points ______________
2. 25 points ______________
3. 25 points ______________
4. 25 points ______________

Total (100 points) ____________

1. **[25 points]** Consider the following continuous-time Markov process $Y = (Y_t)_{t \geq 0}$ where each Y_t takes values in $\{0, 1\}$. The distribution of Y_0 is fixed. Let $N = (N_t)_{t \geq 0}$ be a Poisson process with unit rate, independent of Y_0. The value of Y_t remains constant between successive count times of N. With each new count, it flips with probability 1/2 and stays the same with probability 1/2.

   (a) Show that Y is time-homogeneous, and find its transition matrix $H(t)$ and generator matrix Q.

   (b) Find the invariant distribution of Y (if any).

2. **[25 points]** Let $\mathcal{X}$ and $\mathcal{S}$ be two finite sets, and consider two discrete-time random processes, $X = (X_k)_{k\in\mathbb{Z}_+}$ taking values in $\mathcal{X}$ and $S = (S_k)_{k\in\mathbb{Z}_+}$ taking values in $\mathcal{S}$. The processes are specified as follows. We have a probability distribution μ on $\mathcal{S}$, a collection $(\pi^{(s)} : s \in \mathcal{S})$ of probability distributions on $\mathcal{X}$, and an update function $f : \mathcal{S} \times \mathcal{X} \to \mathcal{S}$. At time $k = 0$, S_0 has distribution μ and X_0 has distribution $\pi^{(S_0)}$. At each subsequent time $k = 1, 2, \ldots$,

$$S_k = f(S_{k-1}, X_{k-1}) \qquad \text{and} \qquad X_k \sim \pi^{(S_k)}.$$

(a) For each $k \geq 0$, let $Y_k = \binom{X_k}{S_k}$. Show that the process $Y = (Y_k)_{k\in\mathbb{Z}_+}$ taking values in $\mathcal{X} \times \mathcal{S}$ and the process $S = (S_k)_{k\in\mathbb{Z}_+}$ are Markov.

(b) Is the process $X = (X_k)_{k\in\mathbb{Z}_+}$ necessarily Markov? Justify your answer. If the answer is 'no,' give a sufficient condition for X to be Markov. (Hint: think about the update function f.)

3. **[25 points]** Let $X = (X_k)_{k\in\mathbb{Z}_+}$ and $Y = (Y_k)_{k\in\mathbb{Z}_+}$ be two discrete-time second-order random processes on a common probability space. We say that Y is a *martingale with respect to* X if the following two conditions hold:

- for each k, Y_k is a function of $X^k = (X_1, \ldots, X_k)$
- for each k, $\mathbb{E}[Y_{k+1}|X^k] = Y_k$

(a) Let $W, X_1, X_2, \ldots$ be second-order random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$. For each k, let $Y_k = \mathbb{E}[W|X^k]$. Prove that $Y = (Y_k)_{k\in\mathbb{Z}_+}$ is a martingale with respect to $X = (X_k)_{k\in\mathbb{Z}_+}$.

(b) Suppose that the random variables $W, X_1, X_2, \ldots$ from part (a) are jointly Gaussian and zero-mean, and that the X_k's are orthogonal. Show that the process Y from part (a) has the recursive representation

$$Y_{k+1} = Y_k + \mathbb{E}[W|X_{k+1}], \qquad k = 0, 1, 2, \ldots$$

4. **[25 points]** Let $X_1, X_2, \ldots$ be a sequence of i.i.d. $N(1,1)$ random variables, and let $N = (N_t)_{t\geq 0}$ be a Poisson process with rate 2, where we assume that N is independent of $X_1, X_2, \ldots$. Define the continuous-time compound process $Y = (Y_t)_{t\geq 0}$ by

$$Y_t = \begin{cases} 0, & t = 0 \\ \sum_{i=1}^{N_t} X_i, & t > 0 \end{cases}$$

(a) Find the mean function of $Y = (Y_t)_{t\geq 0}$.

(b) Show that Y has independent increments.