University of Illinois Fall 2012

ECE 534: Exam I

Wednesday October 24, 2012
6:00 p.m. — 7:30 p.m.
165 Everitt Laboratory

1. **[25 points]** Let X be a random variable taking values in the interval $[0,1]$, and let Y be a random variable jointly distributed with X. We wish to estimate X on the basis of Y.

   (a) In preparation for part (b), prove the *reverse Markov inequality*: If U is a random variable such that $P(U \le c) = 1$ for some $c \in \mathbb{R}$, then for any $u < \mathbb{E}[U]$

   $$P(U > u) \ge \frac{\mathbb{E}[U] - u}{c - u}$$

   (Hint: $V = c - U$ is nonnegative with probability one.)
   Solution: The event $\{U \le u\}$ is equivalent to the event $\{V \ge c - u\}$. Since V is nonnegative, Markov's inequality gives

   $$P\left(U \le u\right) = P\left(V \ge c - u\right) \le \frac{\mathbb{E}[V]}{c-u} = \frac{c - \mathbb{E}[U]}{c - u}$$

   (note that both the numerator and the denominator are strictly positive because $u < \mathbb{E}[U] \le c$). From this,

   $$P\left(U > u\right) = 1 - P\left(U \le u\right) \ge \frac{\mathbb{E}[U] - u}{c - u}.$$

   (b) Let $e = X - \mathbb{E}[X|Y]$ be the error of the MMSE estimator of X from Y. Suppose that the joint distribution of X and Y is such that $\mathbb{E}[e^2] = 1/2$. Use the reverse Markov inequality from part (a) to prove that

   $$P\left(\mathbb{E}[e^2|X] > \frac{1}{4}\right) \ge \frac{1}{3}$$

   Solution: By the tower property of conditional expectations, $\mathbb{E}\left[\mathbb{E}[e^2|X]\right] = \mathbb{E}[e^2] = 1/2$. Since $X \in [0,1]$, $\mathbb{E}[X|Y] \in [0,1]$ as well, which means that $e^2 \in [0,1]$. Applying the reverse Markov inequality with $U = \mathbb{E}[e^2|X]$, $c = 1$ and $u = 1/4$, we get

   $$P\left(\mathbb{E}[e^2|X] > \frac{1}{4}\right) \ge \frac{\mathbb{E}[e^2] - 1/4}{1 - 1/4} = \frac{1/2 - 1/4}{1 - 1/4} = 1/3.$$

2. **[25 points]** A function $f : \mathbb{R} \to \mathbb{R}$ is *c-Lipschitz* for some $c > 0$ if $|f(x) - f(y)| \le c|x - y|$ for all $x, y \in \mathbb{R}$.

   (a) Let $\{X_n\}_{n=1}^{\infty}$ and $\{Z_n\}_{n=1}^{\infty}$ be two sequences of random variables, both converging in the mean square sense. Define a new sequence $\{Y_n\}_{n=1}^{\infty}$ of random variables by

   $$Y_n = f(X_n) + Z_n, \qquad n = 1, 2, \dots$$

   where f is c-Lipschitz. Does the sequence $\{Y_n\}_{n=1}^{\infty}$ converge in the mean-square sense? Justify your answer. (You do not have to identify the limit.)

Solution: Yes, by the Cauchy criterion. For any m, n we have

$$\mathbb{E}\left[(Y_m - Y_n)^2\right] = \mathbb{E}\left[(f(X_m) - f(X_n) + Z_m - Z_n)^2\right]$$
$$= \mathbb{E}\left[(f(X_m) - f(X_n))^2\right] + 2\,\mathbb{E}\left[(f(X_m) - f(X_n))(Z_m - Z_n)\right] + \mathbb{E}\left[(Z_m - Z_n)^2\right].$$

Since f is c-Lipschitz, the first term on the right-hand side is bounded by

$$c^2\,\mathbb{E}\left[(X_m - X_n)^2\right],$$

which converges to zero as $m, n \to \infty$ since X_n converges in mean square sense. The third term likewise converges to zero as $m, n \to \infty$ since Z_n converges in mean square sense. For the middle term, use Cauchy–Schwarz and the Lipschitz property of f:

$$\mathbb{E}\left[(f(X_m) - f(X_n))(Z_m - Z_n)\right] \le \sqrt{\mathbb{E}\left[(f(X_m) - f(X_n))^2\right]\mathbb{E}\left[(Z_m - Z_n)^2\right]}$$

The same argument as before shows that both expectations under the square root converge to zero as $m, n \to \infty$. Therefore, $\mathbb{E}\left[(Y_m - Y_n)^2\right] \to 0$ as $m, n \to \infty$, so the sequence $\{Y_n\}$ converges in the mean square sense.

(b) Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of i.i.d. random variables taking values in the interval $[-1, 1]$, and for each n let $Y_n = f(X_n)$, where f is a 1-Lipschitz function with $\mathbb{E}[f(X_1)] = 0$. Give a direct proof (without using the Strong Law of Large Numbers) that the sequence

$$\overline{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i, \qquad n = 1, 2, \ldots$$

converges almost surely, and identify the limit.

Solution: Each random variable $Y_i = f(X_i)$ is bounded between $\min\limits_{x\in[-1,1]} f(x)$ and $\max\limits_{x\in[-1,1]} f(x)$. Since f is 1-Lipschitz,

$$\max_{x\in[-1,1]} f(x) - \min_{x\in[-1,1]} f(x) \le \max_{x,x'\in[-1,1]} |f(x) - f(x')| \le 2.$$

Moreover, each Y_i has zero mean. Using the Chernoff–Hoeffding bound, for any $\epsilon > 0$ we get

$$P\left(\left|\overline{Y}_n\right| \ge \epsilon\right) \le P\left(\overline{Y}_n \ge \epsilon\right) + P\left(-\overline{Y}_n \ge \epsilon\right) \le 2\exp\left(-\frac{n\epsilon^2}{2}\right).$$

This implies that

$$\sum_{n=1}^{\infty} P\left(\left|\overline{Y}_n\right| \ge \epsilon\right) \le 2\sum_{n=1}^{\infty}\exp\left(-\frac{n\epsilon^2}{2}\right) < \infty, \qquad \forall \epsilon \ge 0.$$

Therefore, by the Borel–Cantelli lemma the sequence $\{\overline{Y}_n\}$ converges to zero almost surely.

3. **[25 points]** Let X, Z_1 and Z_2 be three mutually independent $N(0, 1)$ random variables. Let

$$Y_1 = X + Z_1 + Z_2$$
$$Y_2 = X + 3Z_1 + Z_2$$

(a) Find the simplest expression for $\mathbb{E}[X|Y_1, Y_2]$

Solution: Because X, Z_1 and Z_2 are mutually independent Gaussian random variables, X, Y_1 and Y_2 are jointly Gaussian. Therefore, $\mathbb{E}[X|Y_1, Y_2] = \widehat{\mathbb{E}}[X|Y_1, Y_2]$. Let $Y = (Y_1, Y_2)^T$. Then, because X is zero mean,

$$\widehat{\mathbb{E}}[X|Y] = \text{Cov}(X, Y)\text{Cov}(Y)^{-1}Y,$$

assuming that $\text{Cov}(Y)$ is nonsingular. Using the fact that X, Y_1, Y_2 are independent $N(0, 1)$ random variables, we have

$$\begin{aligned}
\text{Cov}(X, Y) &= (\mathbb{E}[XY_1] \;\; \mathbb{E}[XY_2]) = \left(\mathbb{E}[X^2] \;\; \mathbb{E}[X^2]\right) = (1 \;\; 1) \\
\text{Cov}(Y) &= \begin{pmatrix} \mathbb{E}[Y_1^2] & \mathbb{E}[Y_1Y_2] \\ \mathbb{E}[Y_2Y_1] & \mathbb{E}[Y_2^2] \end{pmatrix} \\
&= \begin{pmatrix} \mathbb{E}[(X + Z_1 + Z_2)^2] & \mathbb{E}[(X + Z_1 + Z_2)(X + 3Z_1 + Z_2)] \\ \mathbb{E}[(X + 3Z_1 + Z_2)(X + Z_1 + Z_2)] & \mathbb{E}[(X + 3Z_1 + Z_2)^2] \end{pmatrix} \\
&= \begin{pmatrix} \mathbb{E}[X^2 + Z_1^2 + Z_2^2] & \mathbb{E}[X^2 + 3Z_1^2 + Z_2^2] \\ \mathbb{E}[X^2 + 3Z_1^2 + Z_2^2] & \mathbb{E}[X^2 + 9Z_1^2 + Z_2^2] \end{pmatrix} \\
&= \begin{pmatrix} 3 & 5 \\ 5 & 11 \end{pmatrix}.
\end{aligned}$$

Indeed, $\det \text{Cov}(Y) = 8$. Therefore,

$$\text{Cov}(Y)^{-1} = \frac{1}{\det \text{Cov}(Y)} \begin{pmatrix} 11 & -5 \\ -5 & 3 \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 11 & -5 \\ -5 & 3 \end{pmatrix}$$

and

$$\begin{aligned}
\mathbb{E}[X|Y_1, Y_2] &= \widehat{\mathbb{E}}[X|Y_1, Y_2] \\
&= \text{Cov}(X, Y)\text{Cov}(Y)^{-1}Y \\
&= \frac{1}{8} \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 11 & -5 \\ -5 & 3 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \\
&= \frac{3Y_1}{4} - \frac{Y_2}{4}
\end{aligned}$$

(b) Find the simplest expression for the corresponding MSE for estimation of X by $\mathbb{E}[X|Y_1, Y_2]$.

Solution: From the solution to part (a),

$$\begin{aligned}
X - \mathbb{E}[X|Y_1, Y_2] &= X - \frac{3Y_1}{4} + \frac{Y_2}{4} \\
&= X - \frac{3}{4}(X + Z_1 + Z_2) + \frac{1}{4}(X + 3Z_1 + Z_2) \\
&= \frac{X}{2} - \frac{Z_2}{2}
\end{aligned}$$

Therefore, $\text{MSE} = \frac{1}{4}\mathbb{E}\left[(X - Z_2)^2\right] = \frac{1}{2}$.

4. **[25 points]** Let X and Y be jointly Gaussian random variables with zero mean, such that the vector $\begin{pmatrix} X \\ Y \end{pmatrix}$ has covariance matrix $\begin{pmatrix} 4 & 4 \\ 4 & 8 \end{pmatrix}$.

(a) Find the conditional expectation $\mathbb{E}[e^{tX}|Y]$. (Your answer should be a function of Y, and it should depend on t as well.)

Solution: For any y, the conditional distribution of X given $Y = y$ is $N(\mathbb{E}[X|Y = y], \mathrm{Var}(e))$, where $e = X - \mathbb{E}[X|Y] = X - \widehat{\mathbb{E}}[X|Y]$. We have

$$\begin{aligned}
\widehat{\mathbb{E}}[X|Y] &= \frac{\mathrm{Cov}(X,Y)Y}{\mathrm{Var}(Y)} = \frac{4Y}{8} = \frac{Y}{2} \\
\mathrm{Var}(e) &= \mathrm{Var}\left(X - \frac{Y}{2}\right) \\
&= \mathrm{Var}(X) - \frac{1}{2}\mathrm{Cov}(X,Y) - \frac{1}{2}\mathrm{Cov}(Y,X) + \frac{1}{4}\mathrm{Var}(Y) \\
&= \mathrm{Var}(X) - \mathrm{Cov}(X,Y) + \frac{1}{4}\mathrm{Var}(Y) = 2.
\end{aligned}$$

Therefore, $\mathbb{E}[e^{tX}|Y = y]$ is equal to the moment generating function of a $N(y/2, 2)$ random variable, which is $e^{ty/2+t^2}$. The answer is $\mathbb{E}[e^{tX}|Y] = e^{tY/2+t^2}$ (note that this is a random variable).

(b) Express the conditional probability $P\big(|X| \geq 4\big|Y\big)$ in terms of the standard Gaussian cdf $\Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-x^2/2} \mathrm{d}x$. (Your answer should be a function of Y.)

Solution: From part (a), the conditional distribution of X given $Y = y$ is $N(y/2, 2)$. If Z has a standard Gaussian distribution $N(0,1)$, then

$$\begin{aligned}
P\big(|X| \geq 4\big|Y = y\big) &= P\big(|\sqrt{2}Z + y/2| \geq 4\big) \\
&= P\big(\sqrt{2}Z \geq 4 - y/2\big) + P\big(\sqrt{2}Z \leq -4 - y/2\big) \\
&= P\left(Z \geq \frac{4 - y/2}{\sqrt{2}}\right) + P\left(Z \leq \frac{-4 - y/2}{\sqrt{2}}\right) \\
&= 1 - \Phi\left(\frac{4 - y/2}{\sqrt{2}}\right) + \Phi\left(\frac{-4 - y/2}{\sqrt{2}}\right)
\end{aligned}$$

Consequently,

$$P\big(|X| \geq 4\big|Y\big) = 1 - \Phi\left(\frac{4 - Y/2}{\sqrt{2}}\right) + \Phi\left(\frac{-4 - Y/2}{\sqrt{2}}\right)$$

(note that this is a random variable).