University of Illinois Fall 2012

ECE 534: Exam I

Wednesday October 24, 2012
6:00 p.m. — 7:30 p.m.
165 Everitt Laboratory

Name: ________________________________

University NetID: ____________________

- You have ninety minutes for this exam. The exam is closed book and closed note, however, you may consult both sides of one $8.5'' \times 11''$ sheet of notes.
- Calculators, laptop computers, smartphones, PDAs, two-way e-mail pagers, etc. may not be used.
- Write your answers in the spaces provided.
- **Please show all of your work. Answers without appropriate justification will receive very little credit.** If you need extra space, use the back of the previous page.
- Useful formula: The inverse A^{-1} of a nonsingular matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is equal to

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Scores

1. 25 points ____________
2. 25 points ____________
3. 25 points ____________
4. 25 points ____________

Total (100 points) ____________

1. **[25 points]** Let X be a random variable taking values in the interval $[0,1]$, and let Y be a random variable jointly distributed with X. We wish to estimate X on the basis of Y.

   (a) In preparation for part (b), prove the *reverse Markov inequality*: If U is a random variable such that $P(U \leq c) = 1$ for some $c \in \mathbb{R}$, then for any $u < \mathbb{E}[U]$

   $$P(U > u) \geq \frac{\mathbb{E}[U] - u}{c - u}$$

   (Hint: $V = c - U$ is nonnegative with probability one.)

   (b) Let $e = X - \mathbb{E}[X|Y]$ be the error of the MMSE estimator of X from Y. Suppose that the joint distribution of X and Y is such that $\mathbb{E}[e^2] = 1/2$. Use the reverse Markov inequality from part (a) to prove that

   $$P\left(\mathbb{E}[e^2|X] > \frac{1}{4}\right) \geq \frac{1}{3}$$

2. **[25 points]** A function $f : \mathbb{R} \to \mathbb{R}$ is *c-Lipschitz* for some $c > 0$ if $|f(x) - f(y)| \leq c|x - y|$ for all $x, y \in \mathbb{R}$.

   (a) Let $\{X_n\}_{n=1}^{\infty}$ and $\{Z_n\}_{n=1}^{\infty}$ be two sequences of random variables, both converging in the mean square sense. Define a new sequence $\{Y_n\}_{n=1}^{\infty}$ of random variables by

   $$Y_n = f(X_n) + Z_n, \qquad n = 1, 2, \ldots$$

   where f is c-Lipschitz. Does the sequence $\{Y_n\}_{n=1}^{\infty}$ converge in the mean-square sense? Justify your answer. (You do not have to identify the limit.)

   (b) Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of i.i.d. random variables taking values in the interval $[-1, 1]$, and for each n let $Y_n = f(X_n)$, where f is a 1-Lipschitz function with $\mathbb{E}[f(X_1)] = 0$. Give a direct proof (without using the Strong Law of Large Numbers) that the sequence

   $$\overline{Y}_n = \frac{1}{n} \sum_{i=1}^{n} Y_i, \qquad n = 1, 2, \ldots$$

   converges almost surely, and identify the limit.

3. **[25 points]** Let X, Z_1 and Z_2 be three mutually independent $N(0,1)$ random variables. Let

$$Y_1 = X + Z_1 + Z_2$$
$$Y_2 = X + 3Z_1 + Z_2$$

(a) Find the simplest expression for $\mathbb{E}[X|Y_1, Y_2]$

(b) Find the simplest expression for the corresponding MSE for estimation of X by $\mathbb{E}[X|Y_1, Y_2]$.

4. **[25 points]** Let X and Y be jointly Gaussian random variables with zero mean, such that the vector $\begin{pmatrix} X \\ Y \end{pmatrix}$ has covariance matrix $\begin{pmatrix} 4 & 4 \\ 4 & 8 \end{pmatrix}$.

   (a) Find the conditional expectation $\mathbb{E}[e^{tX}|Y]$. (Your answer should be a function of Y, and it should depend on t as well.)

   (b) Express the conditional probability $P\big(|X| \geq 4 \big| Y\big)$ in terms of the standard Gaussian cdf $\Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-x^2/2} \mathrm{d}x$. (Your answer should be a function of Y.)