ECE 534: Exam I

Wednesday October 24, 2012
6:00 p.m. — 7:30 p.m.
165 Everitt Laboratory

Name: ________________________________

University NetID: ______________________

• You have ninety minutes for this exam. The exam is closed book and closed note, however, you may consult both sides of one 8.5″ × 11″ sheet of notes.

• Calculators, laptop computers, smartphones, PDAs, two-way e-mail pagers, etc. may not be used.

• Write your answers in the spaces provided.

• Please show all of your work. Answers without appropriate justification will receive very little credit. If you need extra space, use the back of the previous page.

• Useful formula: The inverse $A^{-1}$ of a nonsingular matrix

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

is equal to

\[
A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}
\]

Scores

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Total (100 points) ____________
1. [25 points] Let $X$ be a random variable taking values in the interval $[0,1]$, and let $Y$ be a random variable jointly distributed with $X$. We wish to estimate $X$ on the basis of $Y$.

(a) In preparation for part (b), prove the reverse Markov inequality. If $U$ is a random variable such that $P(U \leq c) = 1$ for some $c \in \mathbb{R}$, then for any $u < \mathbb{E}[U]$

$$P(U > u) \geq \frac{\mathbb{E}[U] - u}{c - u}$$

(Hint: $V = c - U$ is nonnegative with probability one.)

(b) Let $e = X - \mathbb{E}[X|Y]$ be the error of the MMSE estimator of $X$ from $Y$. Suppose that the joint distribution of $X$ and $Y$ is such that $\mathbb{E}[e^2] = 1/2$. Use the reverse Markov inequality from part (a) to prove that

$$P\left(\mathbb{E}[e^2|X] > \frac{1}{4}\right) \geq \frac{1}{3}$$
2. [25 points] A function \( f : \mathbb{R} \to \mathbb{R} \) is \( c \)-Lipschitz for some \( c > 0 \) if \( |f(x) - f(y)| \leq c|x - y| \) for all \( x, y \in \mathbb{R} \).

(a) Let \( \{X_n\}_{n=1}^{\infty} \) and \( \{Z_n\}_{n=1}^{\infty} \) be two sequences of random variables, both converging in the mean square sense. Define a new sequence \( \{Y_n\}_{n=1}^{\infty} \) of random variables by
\[
Y_n = f(X_n) + Z_n, \quad n = 1, 2, \ldots
\]
where \( f \) is \( c \)-Lipschitz. Does the sequence \( \{Y_n\}_{n=1}^{\infty} \) converge in the mean-square sense? Justify your answer. (You do not have to identify the limit.)

(b) Let \( \{X_n\}_{n=1}^{\infty} \) be a sequence of i.i.d. random variables taking values in the interval \([-1, 1]\), and for each \( n \) let \( Y_n = f(X_n) \), where \( f \) is a 1-Lipschitz function with \( \mathbb{E}[f(X_1)] = 0 \). Give a direct proof (without using the Strong Law of Large Numbers) that the sequence
\[
\bar{Y}_n = \frac{1}{n} \sum_{i=1}^{n} Y_i, \quad n = 1, 2, \ldots
\]
converges almost surely, and identify the limit.
3. [25 points] Let $X$, $Z_1$ and $Z_2$ be three mutually independent $N(0,1)$ random variables. Let

$$Y_1 = X + Z_1 + Z_2$$
$$Y_2 = X + 3Z_1 + Z_2$$

(a) Find the simplest expression for $E[X|Y_1, Y_2]$

(b) Find the simplest expression for the corresponding MSE for estimation of $X$ by $E[X|Y_1, Y_2]$. 
4. **[25 points]** Let \( X \) and \( Y \) be jointly Gaussian random variables with zero mean, such that the vector \( \begin{pmatrix} X \\ Y \end{pmatrix} \) has covariance matrix \( \begin{pmatrix} 4 & 4 \\ 4 & 8 \end{pmatrix} \).

(a) Find the conditional expectation \( \mathbb{E}[e^{tX}|Y] \). (Your answer should be a function of \( Y \), and it should depend on \( t \) as well.)

(b) Express the conditional probability \( P(|X| \geq 4|Y) \) in terms of the standard Gaussian cdf \( \Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-x^2/2}dx \). (Your answer should be a function of \( Y \).)