Lecture IX: Fourier transform

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This lecture

Plan for the lecture:

1. Recap: Fourier series representation of periodic signals
2. Frequency content of aperiodic signals: the Fourier transform
3. The inverse Fourier transform
4. Properties of the Fourier transform
5. Generalized Fourier transform
6. Bandlimited and timelimited signals
7. Frequency response of LTI systems

Recap: Fourier series

Recall from the last lecture that any sufficiently regular T-periodic continuous-time signal $x(t)$ can be expanded, e.g., in a complex exponential Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t},$$

where $\omega_0 = 2\pi/T$ is the fundamental frequency, and the Fourier coefficients $\{c_k\}$ are given by

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt, \qquad k = \dots, -2, -1, 0, 1, 2, \dots$$

The Fourier coefficients $\{c_k\}$ tell us about the **frequency content** (or *spectral content*) of $x(t)$.

What about **aperiodic** signals?

Any continuous-time signal $x(t)$ that has finite "energy", i.e.,

$$\int_{-\infty}^{\infty} x^2(t)dt < +\infty,$$

can be represented in the frequency domain via the **Fourier transform**:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

We will also write

$$X(\omega) = \mathcal{F}[x(t)]$$

to denote the fact that $X(\omega)$ is the Fourier transform of $x(t)$.

Example: rectangular pulse

Consider the rectangular pulse

$$p_\tau(t) = \begin{cases} 1, & |t| \le \tau/2 \\ 0, & |t| > \tau/2 \end{cases}$$

$$\begin{aligned}
\mathcal{F}[p_\tau(t)] &= \int_{-\infty}^{\infty} p_\tau(t) e^{-j\omega t} dt \\
&= \int_{-\tau/2}^{\tau/2} e^{-j\omega t} dt \\
&= \left[-\frac{1}{j\omega} e^{-j\omega t} \right]_{-\tau/2}^{\tau/2} \\
&= \frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{j\omega} \\
&= \frac{2\sin(\omega\tau/2)}{\omega} \\
&= \tau \mathrm{sinc}\left(\frac{\tau\omega}{2\pi}\right).
\end{aligned}$$

Inverse Fourier transform

The signal $x(t)$ can be recovered from its Fourier transform $X(\omega) = \mathcal{F}[x(t)]$ using the **inverse Fourier transform** formula

$$x(t) = \mathcal{F}^{-1}[X(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

Note:

- There is a factor of $1/2\pi$ in front of the integral.
- The integration is with respect to ω, for a fixed value of t.

We will also write

$$x(t) \leftrightarrow X(\omega)$$

and say that $x(t)$ [time domain] and $X(\omega)$ [freq. domain] are a **Fourier transform pair**.

Proof:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x(t')e^{-j\omega t'}dt'\right)e^{j\omega t}d\omega$$

$$= \int_{-\infty}^{\infty} x(t')\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j(t-t')\omega}d\omega\right)dt'.$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j(t-t')\omega}d\omega = \frac{1}{2\pi}\lim_{\Omega\to\infty}\int_{-\pi\Omega}^{\pi\Omega} e^{j(t-t')\omega}d\omega$$

$$= \lim_{\Omega\to\infty}\frac{1}{2\pi j(t-t')\Omega}\left[e^{j(t-t')\omega}\right]_{-\pi\Omega}^{\pi\Omega}$$

$$= \lim_{\Omega\to\infty}\frac{\sin(\pi\Omega(t-t'))}{\pi\Omega(t-t')}$$

$$= \lim_{\Omega\to\infty}\operatorname{sinc}(\Omega(t-t'))$$

$$= \delta(t-t')$$

Hence,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega = \int_{-\infty}^{\infty} x(t')\delta(t-t')dt' = x(t)$$

Q.E.D.

Properties of the Fourier transform

The Fourier transform has many useful properties that make calculations easier and also help thinking about the structure of signals and the action of systems on signals.

The properties are listed in any textbook on signals and systems. We will look at and prove a few of them.

Linearity

The Fourier transform is **linear**: if

$$x_1(t) \leftrightarrow X_1(\omega) \qquad \text{and} \qquad x_2(t) \leftrightarrow X_2(\omega),$$

then

$$c_1 x_1(t) + c_2 x_2(t) \leftrightarrow c_1 X_1(\omega) + c_2 X_2(\omega)$$

for any two numbers c_1 and c_2.

Proof: obvious –

$$\mathcal{F}\left[c_1 x_1(t) + c_2 x_2(t)\right] = \int_{-\infty}^{\infty} \left[c_1 x_1(t) + c_2 x_2(t)\right] e^{-j\omega t} dt$$

$$= c_1 \int_{-\infty}^{\infty} x_1(t) e^{-j\omega t} dt + c_2 \int_{-\infty}^{\infty} x_2(t) e^{-j\omega t} dt$$

$$= c_1 X_1(\omega) + c_2 X_2(\omega)$$

Q.E.D.

Time shift

If $x(t) \leftrightarrow X(\omega)$, then $x(t-c) \leftrightarrow X(\omega)e^{-j\omega c}$ for any constant c.

Proof:

$$\begin{aligned} \mathcal{F}[x(t-c)] &= \int_{-\infty}^{\infty} x(t-c)e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} x(t)e^{-j\omega(t+c)}dt \\ &= e^{-j\omega c}\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\ &= X(\omega)e^{-j\omega c}. \end{aligned}$$

Q.E.D.

Multiplication by a complex exponential

If $x(t) \leftrightarrow X(\omega)$, then $x(t)e^{j\omega_0 t} \leftrightarrow X(\omega - \omega_0)$ for any real ω_0.

Proof:

$$\begin{aligned} \mathcal{F}\left[x(t)e^{j\omega_0 t}\right] &= \int_{-\infty}^{\infty} x(t)e^{j\omega_0 t}e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_0)t}dt \\ &= X(\omega - \omega_0). \end{aligned}$$

Q.E.D.

Multiplication by a cosine

If $x(t) \leftrightarrow X(\omega)$, then $x(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}[X(\omega+\omega_0) + X(\omega-\omega_0)]$.

Proof: use linearity and the last property to get

$$\begin{aligned}
\mathcal{F}\left[x(t)\cos(\omega_0 t)\right] &= \mathcal{F}\left[\frac{1}{2}x(t)\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)\right] \\
&= \frac{1}{2}\mathcal{F}\left[x(t)e^{j\omega_0 t}\right] + \frac{1}{2}\mathcal{F}\left[x(t)e^{-j\omega_0 t}\right] \\
&= \frac{1}{2}\left[X(\omega-\omega_0) + X(\omega+\omega_0)\right].
\end{aligned}$$

Q.E.D.

Convolution in time domain

If $x(t) \leftrightarrow X(\omega)$ and $v(t) \leftrightarrow V(\omega)$, then

$$x(t) \star v(t) \leftrightarrow X(\omega)V(\omega)$$

Proof:

$$\begin{aligned}
\mathcal{F}[x(t) \star v(t)] &= \int_{-\infty}^{\infty} [x(t) \star v(t)] e^{-j\omega t} dt \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\lambda) v(t-\lambda) d\lambda \right) e^{-j\omega t} dt \\
&= \int_{-\infty}^{\infty} x(\lambda) \underbrace{\left(\int_{-\infty}^{\infty} v(t-\lambda) e^{-j\omega t} dt \right)}_{\mathcal{F}[v(t-\lambda)]} d\lambda \\
&= \int_{-\infty}^{\infty} x(\lambda) V(\omega) e^{-j\omega\lambda} d\lambda \\
&= V(\omega) \int_{-\infty}^{\infty} x(\lambda) e^{-j\omega\lambda} d\lambda \\
&= X(\omega)V(\omega).
\end{aligned}$$

Q.E.D.

Parseval's theorem

Let $x(t)$ and $v(t)$ be real-valued signals. Then

$$\int_{-\infty}^{\infty} x(t)v(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega)d\omega$$

Proof:

$$\begin{aligned}
\int_{-\infty}^{\infty} x(t)v(t)dt &= \int_{-\infty}^{\infty} x(t)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} V(\omega)e^{j\omega t}d\omega\right)dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} V(\omega)\left(\int_{-\infty}^{\infty} x(t)e^{j\omega t}dt\right)d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} V(\omega)X(-\omega)d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{X(\omega)}V(\omega)d\omega,
\end{aligned}$$

where we used the fact that, since $x(t)$ is real,

$$\overline{X(\omega)} = \int_{-\infty}^{\infty} x(t)e^{j\omega t}dt = X(-\omega).$$

Q.E.D.

Parseval's theorem: cont'd

An important consequence of Parseval's theorem is that

$$\int_{-\infty}^{\infty} x^2(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega.$$

In other words, signal energy can be computed both in time domain and in frequency domain (up to a factor of $1/2\pi$).

Duality

If $x(t) \leftrightarrow X(\omega)$, then $X(t) \leftrightarrow 2\pi x(-\omega)$.

Proof:

$$\begin{aligned}
\mathcal{F}[X(t)] &= \int_{-\infty}^{\infty} X(t) e^{-j\omega t} dt \\
&= 2\pi \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} X(t) e^{-j\omega t} dt \\
&= 2\pi \cdot \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega') e^{-j\omega\omega'} d\omega'}_{=\mathcal{F}^{-1}[X(\omega)](-\omega)} \\
&= 2\pi \cdot \frac{1}{2\pi} x(-\omega)
\end{aligned}$$

Q.E.D.

Duality: an example

Let

$$x(t) = \tau \operatorname{sinc}\left(\frac{\tau t}{2\pi}\right).$$

Then by duality we have

$$X(\omega) = 2\pi p_\tau(\omega).$$

In more detail:

$$p_\tau(t) \leftrightarrow \tau \operatorname{sinc}\left(\frac{\tau \omega}{2\pi}\right)$$

Thus, by duality,

$$\tau \operatorname{sinc}\left(\frac{\tau t}{2\pi}\right) \leftrightarrow 2\pi_\tau(\omega).$$

Generalized Fourier transform

The Fourier transform is defined only for signals with finite energy. However, we can extend its scope by allowing singularity functions.

We begin by computing the Fourier transform of the unit impulse $\delta(t)$.

$$\begin{aligned} \mathcal{F}[\delta(t)] &= \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} \delta(t) dt \\ &= 1, \end{aligned}$$

where we used the sifting property of the unit impulse.

By duality, we have

$$1 \leftrightarrow 2\pi\delta(-\omega) = 2\pi\delta(\omega).$$

Fourier transform of the cosine

The cosine signal $x(t) = \cos(\omega_0 t)$ does not have the Fourier transform in the ordinary sense. It does, however, have a generalized Fourier transform:

$$
\begin{aligned}
\mathcal{F}[\cos(\omega_0 t)] &= \mathcal{F}\left[\frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})\right] \\
&= \frac{1}{2}\mathcal{F}\left[1 \cdot e^{j\omega_0 t}\right] + \frac{1}{2}\mathcal{F}\left[1 \cdot e^{-j\omega_0 t}\right] \\
&= \frac{1}{2}\left[2\pi\delta(\omega - \omega_0) + 2\pi\delta(\omega + \omega_0)\right] \\
&= \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0).
\end{aligned}
$$

Q.E.D.

Fourier transform of a periodic signal

Using the generalized Fourier transform, we can analyze periodic signals that do not have a Fourier transform in the ordinary sense. Thus, if $x(t)$ is a T-periodic signal, we can expand it in a complex exponential Fourier series as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}.$$

$$\begin{aligned} X(\omega) &= \mathcal{F}\left[\sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}\right] \\ &= \sum_{k=-\infty}^{\infty} c_k \mathcal{F}\left[e^{jk\omega_0 t}\right] \\ &= \sum_{k=-\infty}^{\infty} 2\pi c_k \delta(\omega - k\omega_0). \end{aligned}$$

Thus, the (generalized) Fourier transform of a periodic signal is a train of impulses located at integer multiples of the fundamental frequency ω_0.

Bandlimited and timelimited signals

A signal $x(t)$ is called:

- **bandlimited** if there exists a number $B > 0$ (called the **bandwidth**), such that

$$X(\omega) = 0, \qquad \text{for all } |\omega| \geq B.$$

- **timelimited** if there exists a number $T > 0$, such that

$$x(t) = 0, \qquad \text{for all } |t| \geq T.$$

It can be proved that a bandlimited signal cannot be timelimited, and vice versa. We've seen an example of this with the transform pairs

$$p_\tau(t) \leftrightarrow \tau \operatorname{sinc}\left(\frac{\tau\omega}{2\pi}\right) \qquad \text{and} \qquad \tau \operatorname{sinc}\left(\frac{\tau t}{2\pi}\right) \leftrightarrow 2\pi p_\tau(\omega)$$

However, a signal can be approximately timelimited and bandlimited — that is, there exist numbers $B > 0$ and $T > 0$, such that $|x(t)|$ is small for $|t| \geq T$ and $|X(\omega)|$ is small for $|\omega| \geq B$.

Frequency response of LTI systems

Consider an LTI system with the impulse response $h(t)$. Then the output of the system due to input $x(t)$ is given by the convolution integral,

$$y(t) = x(t) \star h(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda.$$

In frequency domain, the action of the system can be described as follows:

$$Y(\omega) = H(\omega)X(\omega).$$

This is a consequence of the fact that convolution in time domain corresponds to multiplication in frequency domain.

The Fourier transform $H(\omega)$ of the impulse response $h(t)$ is called the **frequency response** of the system.