Lecture VIII: Fourier series

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This lecture

Plan for the lecture:

1. Review of vectors and vector spaces
2. Vector space of continuous-time signals
3. Vector space of T-periodic signals
4. Complete orthonormal systems of functions
5. Trigonometric Fourier series
6. Complex exponential Fourier series

Recall vectors in n-space $\mathbb{R}^n$. Each such vector $\mathbf{u}$ can be uniquely represented as a linear combination of n unit vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$:

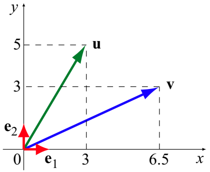
$$\mathbf{u} = \alpha_1 \mathbf{e}_1 + \alpha_2 \mathbf{e}_2 + \ldots + \alpha_n \mathbf{e}_n,$$

where $\alpha_1, \ldots, \alpha_n$ are real numbers (coefficients). These can be computed using the scalar (or dot) product as follows:

$$\alpha_k = \mathbf{u} \cdot \mathbf{e}_k, \qquad k = 1, \ldots, n$$

Note that the vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are:

- normalized — $\mathbf{e}_k \cdot \mathbf{e}_k = 1$ for all k
- orthogonal — $\mathbf{e}_k \cdot \mathbf{e}_m = 0$ for $k \neq m$



Example: $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^2$
$\mathbf{u} = 3\mathbf{e}_1 + 5\mathbf{e}_2$
$\mathbf{v} = 6.5\mathbf{e}_1 + 3\mathbf{e}_2$

Vector spaces

By definition, vectors are objects that can be added together and multiplied by scalars:

- if $\mathbf{u} = \sum_{k=1}^{n} \alpha_k \mathbf{e}_k$ and $\mathbf{v} = \sum_{k=1}^{n} \beta_k \mathbf{e}_k$, then we can form their sum

$$\mathbf{u} + \mathbf{v} = \sum_{k=1}^{n} (\alpha_k + \beta_k) \mathbf{e}_k$$

- if $\mathbf{u} = \sum_{k=1}^{n} \alpha_k \mathbf{e}_k$ and β is a scalar, then we can form the vector

$$\beta \mathbf{u} = \sum_{k=1}^{n} \beta \alpha_k \mathbf{e}_k$$

More generally, a collection of objects that can be added and/or multiplied by scalars is called a **vector space**.

Signal spaces

We have already seen one example of a vector space — the space of continuous-time signals. We can easily verify:

- we can form the sum of any two signals $x_1(t)$ and $x_2(t)$ to get another signal
$$x(t) = x_1(t) + x_2(t)$$
- we can multiply any signal $x(t)$ by a constant c to get another signal
$$z(t) = cx(t)$$

Unlike the n-space $\mathbb{R}^n$, the vector space of all continuous-time signals is **huge**. In fact, it is infinite-dimensional.

The space of periodic signals

Let us consider all T-periodic signals. Any such signal $x(t)$ satisfies

$$x(t+T) = x(t) \qquad \text{for all } t$$

for some given $T > 0$.

It is easy to see that T-periodic signals form a vector space:

- if $x_1(t)$ and $x_2(t)$ are T-periodic, then

  $$x_1(t+T) + x_2(t+T) = x_1(t) + x_2(t),$$

  so their sum is T-periodic
- if $x(t)$ is T-periodic, then

  $$cx(t+T) = cx(t),$$

  so any scaled version of $x(t)$ is also T-periodic

If we impose even more conditions on our T-periodic signals (the so-called Dirichlet conditions, which hold for all signals encountered in practice), then we can represent signals as infinite linear combinations of orthogonal and normalized (orthonormal) vectors.

Complete orthonormal sets of functions

First of all, we can define a scalar (or dot) product of two T-periodic signals $x_1(t)$ and $x_2(t)$ as

$$\langle x_1, x_2 \rangle = \int_0^T x_1(t) x_2(t) dt$$

(note that we can integrate over any whole period, not necessarily from $t = 0$ to $t = T$).

Then we can take any sequence of T-periodic functions (signals) $\phi_0(t), \phi_1(t), \phi_2(t), \ldots$ that are

1. **normalized:** $\langle \phi_k, \phi_k \rangle = \int_0^T \phi_k^2(t) dt = 1$
2. **orthogonal:** $\langle \phi_k, \phi_m \rangle = \int_0^T \phi_k(t) \phi_m(t) dt = 0$ if $k \neq m$
3. **complete:** if a signal $x(t)$ is such that
   $\langle x, \phi_k \rangle = \int_0^T x(t) \phi_k(t) dt = 0$ for all k, then $x(t) \equiv 0$

Let $\{\phi_k(t)\}_{k=1}^{\infty}$ be a complete, orthonormal set of functions. Then any "well-behaved" T-periodic signal $x(t)$ can be uniquely represented as an infinite series

$$x(t) = \sum_{k=0}^{\infty} \alpha_k \phi_k(t).$$

This is called the **Fourier series** representation of $x(t)$. The scalars (numbers) α_k are called the **Fourier coefficients** of $x(t)$ (with respect to $\{\phi_k(t)\}$) and are computed as follows:

$$\alpha_k = \langle x, \phi_k \rangle = \int_0^T x(t)\phi_k(t)dt$$

In analogy to vectors in n-space, you can think of α_k as the projection (or component) of $x(t)$ in the direction of $\phi_k(t)$.

Fourier coefficients

To derive the formula for α_k, write

$$x(t)\phi_k(t) = \sum_{m=0}^{\infty} \alpha_m \phi_m(t)\phi_k(t),$$

and integrate over one period:

$$\begin{aligned}
\underbrace{\int_0^T x(t)\phi_k(t)dt}_{\langle x,\phi_k\rangle} &= \int_0^T \left(\sum_{m=0}^{\infty} \alpha_m \phi_m(t)\phi_k(t)\right) dt \\
&= \sum_{m=0}^{\infty} \alpha_m \underbrace{\int_0^T \phi_m(t)\phi_k(t)dt}_{\langle \phi_m,\phi_k\rangle} \\
&= \alpha_k,
\end{aligned}$$

where in the last line we use the fact that $\{\phi_k\}$ form an orthonormal system of functions.

Convergence of Fourier series

It can be proved that, because the functions $\{\phi_k\}$ form a complete orthonormal system, the partial sums of the Fourier series

$$x(t) = \sum_{k=0}^{\infty} \alpha_k \phi_k(t)$$

converge to $x(t)$ in the following sense:

$$\lim_{N\to\infty} \int_0^T \left(x(t) - \sum_{k=1}^{N} \alpha_k \phi_k(t) \right)^2 dt = 0$$

So, we can use the partial sums

$$x_N(t) = \sum_{k=1}^{N} \alpha_k \phi_k(t)$$

to approximate $x(t)$.

Trigonometric Fourier series

Fact: the sequence of T-periodic functions $\{\phi_k(t)\}_{k=0}^{\infty}$ defined by

$$\phi_0(t) = \frac{1}{\sqrt{T}} \quad \text{and} \quad \phi_k(t) = \begin{cases} \sqrt{\frac{2}{T}} \sin(k\omega_0 t), & \text{if } k \ge 1 \text{ is odd} \\ \sqrt{\frac{2}{T}} \cos(k\omega_0 t), & \text{if } k > 1 \text{ is even} \end{cases}$$

is complete and orthonormal. Here,

$$\omega_0 = \frac{2\pi}{T}$$

is called the **fundamental frequency**. Orthonormality is quite easy to show, completeness — not so much.

Thus, any well-behaved T-periodic signal $x(t)$ can be represented as an infinite sum of sinusoids (plus a constant term $\alpha_0\phi_0$):

$$x(t) = \sum_{k=0}^{\infty} \alpha_k \phi_k(t)$$

Trigonometric Fourier series

A more common way of writing down the trigonometric Fourier series of $x(t)$ is this:

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t)$$

Then the Fourier coefficients can be computed as follows:

$$\begin{aligned} a_0 &= \frac{1}{T}\int_0^T x(t)dt \\ a_k &= \frac{2}{T}\int_0^T x(t)\cos(k\omega_0 t)dt \\ b_k &= \frac{2}{T}\int_0^T x(t)\sin(k\omega_0 t)dt \end{aligned}$$

Recall that $\omega_0 = 2\pi/T$.

To relate this to the orthonormal representation in terms of the ϕ_k, we note that we can write

$$a_0 = \frac{1}{\sqrt{T}} \int_0^T x(t)\phi_0(t)dt = \frac{1}{\sqrt{T}}\alpha_0$$

$$a_k = \sqrt{\frac{2}{T}} \int_0^T x(t)\phi_{2k}(t)dt = \sqrt{\frac{2}{T}}\alpha_{2k}$$

$$b_k = \sqrt{\frac{2}{T}} \int_0^T x(t)\phi_{2k-1}(t)dt = \sqrt{\frac{2}{T}}\alpha_{2k-1}$$

Thus, we have

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t)$$

$$= \frac{1}{\sqrt{T}}\alpha_k \cdot \sqrt{T}\phi_0(t) + \sqrt{\frac{2}{T}} \left(\sum_{k=1}^{\infty} \alpha_{2k} \cdot \sqrt{\frac{T}{2}}\phi_{2k}(t) + \sum_{k=1}^{\infty} \alpha_{2k-1} \cdot \sqrt{\frac{T}{2}}\phi_{2k-1}(t) \right)$$

$$= \sum_{k=1}^{\infty} \alpha_k \phi_k(t)$$

Symmetry properties

Things to watch out for when computing the Fourier coefficients:

- if $x(t)$ is an even function, i.e., $x(t) = x(-t)$ for all t, then all its sine Fourier coefficients are zero:

$$b_k = \frac{2}{T} \int_{-2/T}^{T/2} x(t) \sin(k\omega_0 t) dt = 0$$

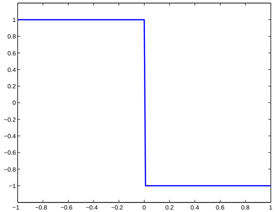
- if $x(t)$ is an odd function, i.e., $x(t) = -x(-t)$, then all its cosine Fourier coefficients are zero:

$$a_k = \frac{2}{T} \int_{-2/T}^{T/2} x(t) \cos(k\omega_0 t) dt = 0,$$

  and

$$a_0 = \frac{1}{T} \int_{-2/T}^{T/2} x(t) dt = 0$$

Consider a 2-periodic signal $x(t)$ given by repeating the square wave:



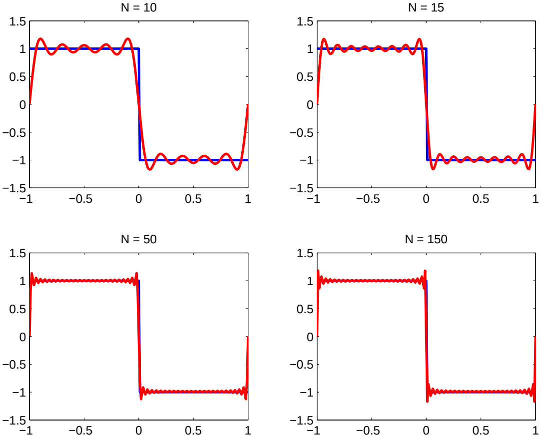
$a_0 = 0$ and $a_k = 0$ for all k (the signal has odd symmetry)

$$
\begin{aligned}
b_k &= \int_{-1}^{0} \sin(k\pi t)dt - \int_{0}^{1} \sin(k\pi t)dt \\
&= -\frac{1}{k\pi}\left[\cos(k\pi t)\right]_{-1}^{0} + \frac{1}{k\pi}\left[\cos(k\pi t)\right]_{0}^{1} \\
&= \frac{1}{k\pi}\Big(\cos(-k\pi) - 1 + \cos(k\pi) - 1\Big) \\
&= \frac{1}{k\pi}(2\cos(k\pi) - 2) \\
&= -\frac{4\sin^2(k\pi/2)}{k\pi}
\end{aligned}
$$

$$
x(t) = -\sum_{k=1}^{\infty} \frac{4\sin^2(k\pi/2)}{k\pi}\sin(k\pi t) = -\sum_{k \text{ odd}}^{\infty} \frac{4}{k\pi}\sin(k\pi t)
$$

Trigonometric Fourier series: example

Approximating $x(t)$ by partial sums of its Fourier series:



Note the **Gibbs phenomenon**: the Fourier series (over/under)shoots the actual value of $x(t)$ at points of discontinuity. In signal processing, this effect is also called ringing.

Complex exponential Fourier series

Another useful complete orthonormal set is furnished by the complex exponentials:

$$\phi_k(t) = \frac{1}{\sqrt{T}} e^{jk\omega_0 t}, \qquad k = \dots, -2, -1, 0, 1, 2, \dots$$

where $\omega_0 = 2\pi/T$, as before.

Note that these functions are complex-valued, and we need to redefine the dot product as

$$\langle x_1, x_2 \rangle = \int_0^T x_1(t) x_2^*(t) dt,$$

where $x_1^*(t)$ denotes the complex conjugate of $x_2(t)$. Then it is straightforward to show that

$$\langle \phi_k, \phi_m \rangle = \frac{1}{T} \int_0^T e^{jk\omega_0 t} e^{-jm\omega_0 t} dt = \begin{cases} 1, & k = m \\ 0, & k \neq m \end{cases}$$

Complex exponential Fourier series

Thus, we can expand any T-periodic $x(t)$ as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}$$

The Fourier coefficients are given by

$$c_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$$

To derive this, multiply the series representation of $x(t)$ on the right by $e^{-jk\omega_0 t}$ and integrate from 0 to T.

Symmetry properties for real signals

Consider a **real-valued** T-periodic signal $x(t)$. Then

$$c_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt \quad \text{and} \quad c_{-k} = \frac{1}{T}\int_0^T x(t)e^{jk\omega_0 t}dt = c_k^*$$

Write c_k in polar form:

$$c_k = A_k e^{j\theta_k}, \qquad A_k = |c_k|, \theta_k = \angle c_k$$

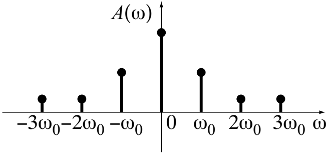
Then

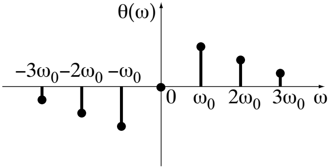
$$c_k = c_{-k}^* \quad \Rightarrow \quad A_k = A_{-k} \text{ and } \angle c_k = -\angle c_{-k}$$

Thus, for real signals the **amplitude spectrum** A_k has even symmetry, while the **phase spectrum** θ_k has odd symmetry.

Amplitude and phase spectra

We can use the amplitudes A_k and the phases θ_k to represent the **spectrum** (or **frequency content**) of $x(t)$ graphically as follows:





Observe that the **amplitude spectrum** $A(\omega)$ has even symmetry, while the **phase spectrum** $\theta(\omega)$ has odd symmetry.