Lecture VI: Convolution representation of discrete-time systems

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This lecture

Plan for the lecture:

1. The unit pulse response
2. The convolution representation of discrete-time LTI systems
3. Convolution of discrete-time signals
4. Causal LTI systems with causal inputs
5. Discrete convolution: an example

The unit pulse response

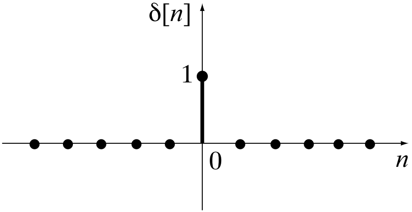
Let us consider a discrete-time LTI system

$$y[n] = S\Big\{x[n]\Big\}$$

and use the **unit pulse**

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

as input.



Let us define the **unit pulse response** of S as the corresponding output:

$$h[n] \triangleq S\Big\{\delta[n]\Big\}$$

We will now show that the output of S due to an **arbitrary** input $x[n]$ can be expressed in terms of $x[n]$ and $h[n]$.

The unit pulse response: cont'd

Consider an arbitrary discrete-time signal $x[n]$. We can write

$$x[n] = \sum_{i=-\infty}^{\infty} x[i]\delta[n-i]$$

(follows by definition of the unit pulse). Then

$$\begin{aligned} y[n] &= S\Big\{x[n]\Big\} \\ &= S\Big\{\sum_{i=-\infty}^{\infty} x[i]\delta[n-i]\Big\} \\ &= \sum_{i=-\infty}^{\infty} S\Big\{x[i]\delta[n-i]\Big\} \quad \text{(because } S \text{ is additive)} \\ &= \sum_{i=-\infty}^{\infty} x[i]S\Big\{\delta[n-i]\Big\} \quad \text{(because } S \text{ is homogeneous)} \end{aligned}$$

The convolution representation

Now, because S is time-invariant, for each i we have

$$S\left\{\delta[n-i]\right\} = h[n-i].$$

Hence,

$$y[n] = \sum_{i=-\infty}^{\infty} x[i]h[n-i],$$

where $h[n]$ is the **unit pulse response** of S. This is known as the **convolution representation** of a discrete-time LTI system.

This name comes from the fact that a summation of the above form is known as the **convolution** of two signals, in this case $x[n]$ and $h[n] = S\left\{\delta[n]\right\}$.

Convolution of discrete-time signals

Let $x[n]$ and $\nu[n]$ be two discrete-time signals. Then their **convolution** is defined as

$$x[n] \star \nu[n] \triangleq \sum_{i=-\infty}^{\infty} x[i]\nu[n-i]$$

(here i is a dummy index).

Thus, if h is the unit pulse response of an LTI system S, then we can write

$$y[n] = S\Big\{x[n]\Big\} = x[n] \star h[n]$$

for any input signal $x[n]$.

Useful properties of convolutions

The convolution operation $\star$ is

1. **commutative:** $x[n] \star \nu[n] = \nu[n] \star x[n]$
2. **associative:** $x[n] \star (\nu[n] \star \mu[n]) = (x[n] \star \nu[n]) \star \mu[n]$
3. **distributive:** $x[n] \star (\nu[n] + \mu[n]) = x[n] \star \nu[n] + x[n] \star \mu[n]$
4. **homogeneous:** $x[n] \star (a\nu[n]) = a(x[n] \star \nu[n])$, where a is any constant

All of these properties are easy to prove from the definition of convolution.

Convolution representation: recap

What we have shown is the following:

1. Any discrete-time LTI system S is uniquely described by its **unit pulse response** $h[n]$, which is defined as the output of the system due to the unit pulse input:

$$h[n] = S\Big\{\delta[n]\Big\}$$

2. The output of S due to an arbitrary input $x[n]$ is given by the **convolution** of $x[n]$ with the unit pulse response $h[n]$:

$$y[n] = S\Big\{x[n]\Big\} = x[n] \star h[n]$$

Unit pulse response of a causal LTI system

Consider a causal LTI system S. Recall that a system is **causal** if and only if its output at time n does not depend on the input at times $m > n$.

Claim 1: if $h[n]$ is the unit pulse response of a causal LTI system, then

$$h[n] = 0 \qquad \text{for all } n < 0.$$

Proof: consider some input signal $x[n]$. Then we have

$$\begin{aligned} S\Big\{x[n]\Big\} &= h[n] \star x[n] \\ &= \sum_{i=-\infty}^{\infty} x[n-i]h[i] \\ &= \underbrace{\sum_{i=-\infty}^{-1} x[n-i]h[i]}_{\text{dep. on future inputs}} + \sum_{i=0}^{\infty} x[n-i]h[i]. \end{aligned}$$

Because the input $x[n]$ is arbitrary, we see that the system will not be causal if $h[i] \neq 0$ for at least one $i < 0$.

Unit pulse response of a causal LTI system

Claim 2: if the unit pulse response $h[n]$ of an LTI system S satisfies

$$h[n] = 0, \qquad \text{for all } n < 0,$$

then S is causal.

Proof: consider some input signal $x[n]$. Then

$$\begin{aligned} S\Big\{x[n]\Big\} &= \sum_{i=-\infty}^{\infty} x[n-i]h[i] \\ &= \underbrace{\sum_{i=-\infty}^{-1} x[n-i]h[i]}_{=0} + \sum_{i=0}^{\infty} x[n-i]h[i] \\ &= \sum_{i=0}^{\infty} x[n-i]h[i]. \end{aligned}$$

Thus, the output at time n depends only on the input at times $n-i$, $i \geq 0$, so the system is causal.

Unit pulse response of a causal LTI system

When dealing with convolutions, you have to be careful with upper and lower limits of sums. In particular, if $h[n]$ is a unit pulse response of a causal LTI system, then $h[n] = 0$ for all $n < 0$. Therefore,

$$x[n] \star h[n] = \sum_{i=0}^{\infty} x[n-i]h[i].$$

Let us now change the dummy index to $k = n - i$. Then we have

$$x[n] \star h[n] = \sum_{k=-\infty}^{n} x[k]h[n-k] = \sum_{i=-\infty}^{n} x[i]h[n-i].$$

Causal LTI systems with causal inputs

We have seen that the unit pulse response $h[n]$ of a causal LTI system is equal to zero for all $n < 0$. Motivated by this observation, let us say that a signal $x[n]$ is **causal** if $x[n] = 0$ for all $n < 0$. Thus, an LTI system is causal if and only if its unit pulse response is a causal signal.

Now consider a causal LTI system S with unit pulse response $h[n]$ and a causal input $x[n]$. Then we have

$$\begin{aligned} x[n] \star h[n] &= \sum_{i=-\infty}^{n} x[i]h[n-i] \\ &= \underbrace{\sum_{i=-\infty}^{-1} x[i]h[n-i]}_{=0} + \sum_{i=0}^{n} x[i]h[n-i]. \end{aligned}$$

Thus, for a causal LTI system with a causal input, the convolution sum that gives the output at time n runs only from $i = 0$ to $i = n$.

Computing discrete convolutions: flip and shift

When computing a convolution $x[n] \star \nu[n]$ of two signals, we note that the convolution sum

$$x]n] \star \nu[n] = \sum_{i=-\infty}^{\infty} x[i]\nu[n-i]$$

gives the "overlap" between the original signal $x[i]$ and the flipped-and-shifted signal $\nu[n-i]$. Since convolution is commutative, we can write

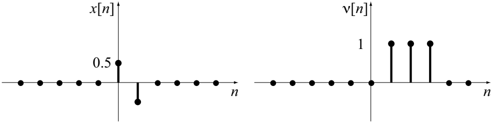
$$\nu[n] \star x[n] = \sum_{i=-\infty}^{\infty} x[n-i]\nu[i],$$

which is the "overlap" between the original signal $\nu[i]$ and the flipped-and-shifted signal $x[n-i]$.

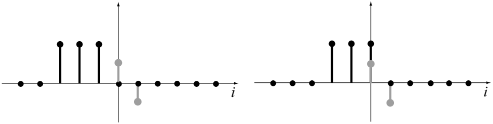
Computing discrete convolutions: an example

When the signals $x[n]$ and $\nu[n]$ have only finitely many nonzero values, the convolution can be computed graphically. In that case, you should flip and shift the "simpler" of the two signals.

Let us show how to convolve the signals $x[n]$ and $\nu[n]$ shown below:

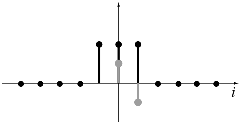


We will flip and shift the simpler signal, which in this case is $\nu[n]$:

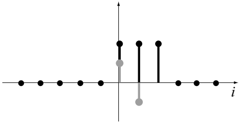


$n = 0$: overlap = 0 $\qquad$ $n = 1$: overlap = 0.5

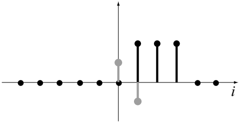
Computing discrete convolutions: an example, cont'd



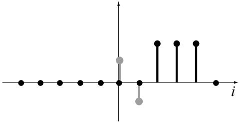
$n = 2$: overlap = 0



$n = 3$: overlap = 0



$n = 4$: overlap = -0.5



$n = 5$: overlap = 0

So, the convolution $y[n] = x[n] \star \nu[n]$ looks like this: