# Lecture IV: LTI models of physical systems

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# This lecture

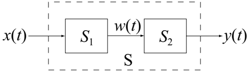
Plan for the lecture:

1. Interconnections of linear systems
2. Differential equation models of LTI systems
3. Review of linear circuit theory
   - resistors, inductors, capacitors
   - Kirchhoff's laws
4. Examples of RLC circuits
5. Leaky integrate-and-fire (LIF) neuron

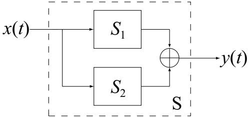
# Interconnections of linear systems

Linearity is preserved when systems are interconnected.

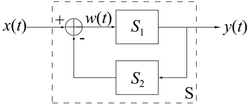
- cascade: $S\left\{x(t)\right\} = S_2\left\{S_1\{x(t)\}\right\}$



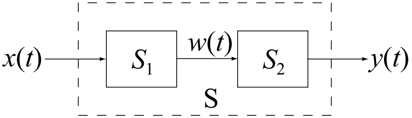

- sum: $S\left\{x(t)\right\} = S_1\left\{x(t)\right\} + S_2\left\{x(t)\right\}$




- feedback $S\left\{x(t)\right\} = S_1\left\{x(t) - S_2\{y(t)\}\right\}$



# Cascade




Let $w(t)$ be the output of $S_1$. Then we first use linearity of $S_1$:

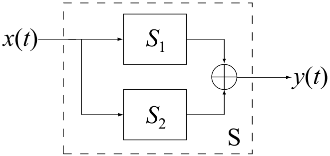
$$w(t) = S_1\Big\{a_1 x_1(t) + a_2 x_2(t)\Big\} = a_1 S_1\Big\{x_1(t)\Big\} + a_2 S_1\Big\{x_2(t)\Big\}$$

Now use linearity of $S_2$:

$$\begin{aligned} S\Big\{a_1 x_1(t) + a_2 x_2(t)\Big\} &= S_2\Big\{w(t)\Big\} = S_2\Big\{a_1 S_1\{x_1(t)\} + a_2 S_1\{x_2(t)\}\Big\} \\ &= a_1 S_2\Big\{S_1\{x_1(t)\}\Big\} + a_2 S_2\Big\{S_1\{x_2(t)\}\Big\} \\ &= a_1 S\Big\{x_1(t)\Big\} + a_2 S\Big\{x_2(t)\Big\} \end{aligned}$$

This proves that $S$ is linear.

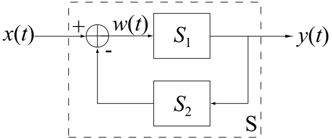
# Sum




$$
\begin{aligned}
S\Big\{a_1x_1(t) + a_2x_2(t)\Big\} \\
&= S_1\Big\{a_1x_1(t) + a_2x_2(t)\Big\} + S_2\Big\{a_1x_1(t) + a_2x_2(t)\Big\} \\
&= \underbrace{a_1S_1\{x_1(t)\} + a_2S_1\{x_2(t)\}}_{\text{use linearity of } S_1} + \underbrace{a_1S_2\{x_1(t)\} + a_2S_2\{x_2(t)\}}_{\text{use linearity of } S_2} \\
&= a_1\underbrace{\Big(S_1\{x_1(t)\} + S_2\{x_1(t)\}\Big)}_{=S\{x_1(t)\}} + a_2\underbrace{\Big(S_1\{x_2(t)\} + S_2\{x_2(t)\}\Big)}_{=S\{x_2(t)\}} \\
&= a_1S\{x_1(t)\} + a_2S\{x_2(t)\}
\end{aligned}
$$

This proves that $S$ is linear.

# Feedback




Let $w(t) = x(t) - S_2\{y(t)\}$. Now, $y(t) = S_1\{w(t)\}$, so

$$w(t) = x(t) - S_2\Big\{S_1\{w(t)\}\Big\} = x(t) - S\Big\{w(t)\Big\},$$

where $S$ is the cascade of $S_1$ and $S_2$, which is linear if both $S_1$ and $S_2$ are. The system $S_3$ with input $x(t)$ and output $w(t)$, defined by

$$w(t) = x(t) - S\Big\{w(t)\Big\},$$

is linear. Thus,

$$y(t) = S_1\Big\{w(t)\Big\} = S_1\Big\{S_3\{x(t)\}\Big\}$$

is a cascade of $S_3$ and $S_1$, and so is linear.

# LTI systems via differential equations

A lot of continuous-time LTI systems are described by linear differential equations with constant coefficients:

$$\sum_{m=0}^{M} a_m \frac{d^m y(t)}{dt^m} = \sum_{n=0}^{N} b_n \frac{d^n x(t)}{dt^n}$$

where the coefficients $\{a_m\}_{m=1}^{M}$ and $\{b_n\}_{N=1}^{N}$ are independent of $t$.

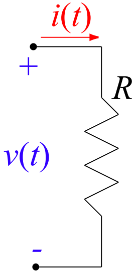
Examples:

- linear electric circuits (RLC)
- mechanical systems (mass-spring-damper)

We will focus on electrical circuits.

# Review: linear circuit elements

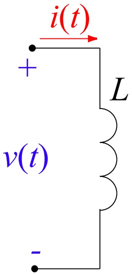
**Resistor:**




$$v(t) = Ri(t)$$

$$i(t) = \frac{v(t)}{R}$$

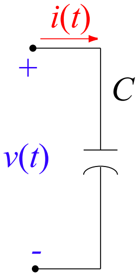
**Inductor:**




$$v(t) = L\frac{di(t)}{dt}$$

$$i(t) = \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau$$

**Capacitor:**

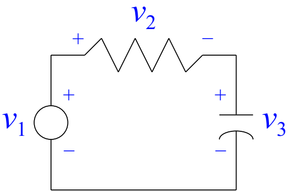



$$i(t) = C\frac{dv(t)}{dt}$$

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$$

# Review: Kirchhoff's laws

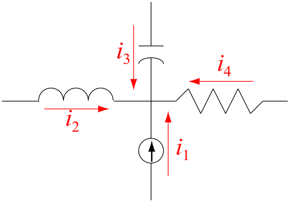
**Kirchhoff's voltage law** (KVL):




The sum of voltages in a loop is equal to zero:

$$-v_1 + v_2 + v_3 = 0$$

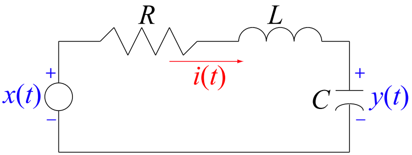
**Kirchhoff's current law** (KCL):




The sum of currents entering a node is equal to zero:

$$i_1 + i_2 + i_3 + i_4 = 0$$

# Example: Series RLC circuit




Input: voltage source $x(t)$
Output: voltage across the capacitor $y(t)$

Apply KVL:

$$-x(t) + Ri(t) + L\frac{di(t)}{dt} + y(t) = 0$$

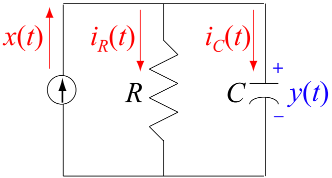
Substitute $i(t) = C\frac{dy(t)}{dt}$:

$$-x(t) + RC\frac{dy(t)}{dt} + LC\frac{d^2y(t)}{dt^2} + y(t) = 0$$

Rearrange to get

$$LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) = x(t)$$

# Example: Parallel RC circuit




Input: current source $x(t)$
Output: voltage across the capacitor $y(t)$

Apply KCL:

$$x(t) = i_R(t) + i_C(t)$$

Substitute $i_R(t) = \frac{y(t)}{R}$ and $i_C(t) = C\frac{dy(t)}{dt}$:

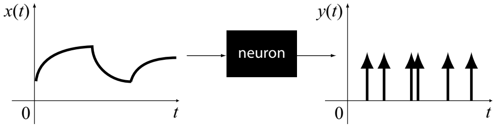
$$x(t) = \frac{y(t)}{R} + C\frac{dy(t)}{dt}$$

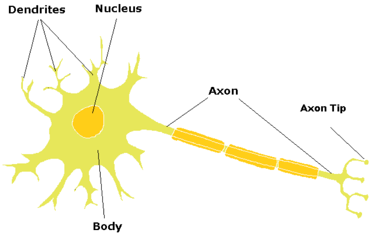
Rearrange to get

$$C\frac{dy(t)}{dt} + \frac{1}{R}y(t) = x(t)$$

# Example: biological neurons

Biological neurons are highly nonlinear systems that convert incoming electrical signals (encoding external stimuli) into spike trains:




Inputs to the neuron are electrical signals traveling along the **dendrites** to the body (or **soma**) of the neuron. The neuron accumulates a potential (voltage) across its cell membrane and then **fires**, i.e., emits an electric pulse that travels down the **axon**.

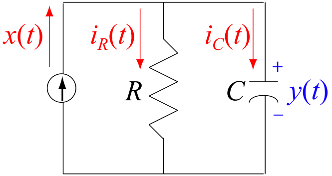
# Leaky integrate-and-fire (LIF) neuron

The **leaky integrate-and-fire (LIF)** neuron is a simple model that describes the salient features of biological neurons. The LIF neuron has two distinct operating regimes:

- subthreshold — when the membrane potential of the neuron is below a certain threshold value $V_{\rm th}$, the neuron acts like a parallel RC circuit. The capacitance is due to charge buildup on both sides of the bilipid layer that forms the cell membrane; the resistance is due to the presence of protein channels in the membrane that can carry $\rm K^+$, $\rm Na^+$ and $\rm Cl^-$ ions in and out of the cell (leakage current)
- superthreshold — when the membrane potential crosses $V_{\rm th}$, the neuron "fires" (emits a unit impulse), and then short-circuits for $\tau_{\rm ref}$ seconds (the time known as the **refractory period**). After the refractory period elapses, the neuron returns to the subthreshold regime.

# Circuit model of the subthreshold regime

Let us look at the subthreshold regime of the LIF neuron with a unit step input $x(t) = u(t)$




$$C\frac{dy(t)}{dt} + \frac{1}{R}y(t) = u(t)$$

$$y(t) = R\left[1 - e^{-t/RC}\right]u(t)$$

The overall output of the LIF neuron due to the unit step input looks like this: