Lecture III: Systems and their properties

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This lecture

Plan for the lecture:

1. What is a system?
2. System properties:
   - causality
   - memory
   - linearity
   - time invariance
3. Linear time-invariant (LTI) systems
4. Nonlinear systems

What is a system?

Recap: a system is any physical device, process or computer algorithm that transforms input signals into output signals.

Examples:

- electronic circuits
- biological systems: audiovisual system, cardiovascular system, etc.
- socioeconomic systems: the stock market, social networks, etc.
- signal processors in scientific or medical equipment or in audio/video devices

We will state our definitions for continuous-time systems. They are essentially the same for discrete-time systems.

Causality

A system S is **causal** if the output at time t does not depend on the values of the input at any time $t' > t$.

Examples

1. Ideal predictor: $y(t) = x(t+1)$ — noncausal since the output at time t depends on the input at future time $t+1$
2. Ideal delay: $y(t) = x(t-1)$ — causal since the output at time t depends only on the input at past time $t-1$
3. Moving average (MA) filter: $y[n] = \frac{x[n-1]+x[n]+x[n+1]}{3}$ — not causal, since the output at time n depends in part on the input at future time $n+1$

Most physical systems are causal. However, noncausal systems are widely used in signal processing, for example, for smoothing of continuous-time and discrete-time signals for noise removal or quality enhancement.

Memory

A causal system S is *memoryless* if the output at time t depends only on the input at time t. Otherwise, the system is said to have memory.

Note: depending on whom you ask, it may or may not make sense to talk about memory for noncausal systems. To avoid confusion, in this class we will only talk about memory for causal systems.

Examples

1. Ideal amplifier: $y(t) = Kx(t)$, where $K > 0$ is the amplifier gain — memoryless, since the output at time t depends only on the input at time t
2. Integrator:

$$y(t) = \int_{-\infty}^{t} x(\tau) d\tau$$

— has memory, since the output at time t depends on the input for all $-\infty < \tau \leq t$.

Linearity

A system S is

- **additive** if for any two inputs $x_1(t)$ and $x_2(t)$,

$$S\Big\{x_1(t) + x_2(t)\Big\} = S\Big\{x_1(t)\Big\} + S\Big\{x_2(t)\Big\}$$

- **homogeneous** if, for any input $x(t)$ and any number a,

$$S\Big\{ax(t)\Big\} = aS\Big\{x(t)\Big\}.$$

A system that is both additive and homogeneous is called **linear**. In other words, S is linear if, for any two inputs $x_1(t)$ and $x_2(t)$ and any two numbers a_1 and a_2,

$$S\Big\{a_1x_1(t) + a_2x_2(t)\Big\} = a_1S\Big\{x_1(t)\Big\} + a_2S\Big\{x_2(t)\Big\}$$

Linearity: Example 1

Suppose the input and the output are related by the differential equation

$$\frac{dy(t)}{dt} = x(t).$$

Additive? Yes:

$$\frac{dy_1(t)}{dt} = x_1(t), \frac{dy_2(t)}{dt} = x_2(t)$$

$$y(t) = S\Big\{x_1(t)+x_2(t)\Big\} \quad \Rightarrow \quad \frac{dy(t)}{dt} = x_1(t)+x_2(t) = \frac{d\big(y_1(t)+y_2(t)\big)}{dt}$$

Homogeneous? Yes:

$$\frac{dy(t)}{dt} = x(t)$$

$$y_a(t) = S\Big\{ax(t)\Big\} \quad \Rightarrow \quad \frac{dy_a(t)}{dt} = ax(t) = a\frac{dy(t)}{dt} = \frac{d\big(ay(t)\big)}{dt}$$

The system is linear.

Linearity: Example 2

Now consider the following system:

$$y(t) = t^2 x(t)$$

Additive? Yes:

$$\begin{aligned} S\Big\{x_1(t) + x_2(t)\Big\} &= t^2\Big(x_1(t) + x_2(t)\Big) \\ &= t^2 x_1(t) + t^2 x_2(t) \\ &= S\Big\{x_1(t)\Big\} + S\Big\{x_2(t)\Big\} \end{aligned}$$

Homogeneous? Yes:

$$S\Big\{ax(t)\Big\} = t^2 ax(t) = at^2 x(t) = aS\Big\{x(t)\Big\}$$

The system is linear.

Linearity: Example 3

Consider the **square-law device**:

$$y(t) = x^2(t)$$

Additive? No:

$$\Big(x_1(t) + x_2(t)\Big)^2 = x_1^2(t) + 2x_1(t)x_2(t) + x_2^2(t) \neq x_1^2(t) + x_2^2(t)$$

So,

$$S\Big\{x_1(t) + x_2(t)\Big\} \neq S\Big\{x_1(t)\Big\} + S\Big\{x_2(t)\Big\}$$

Homogeneous? No:

$$\Big(ax(t)\Big)^2 = a^2x^2(t) \neq ax^2(t) \qquad \text{unless } a = 1$$

The system is nonlinear.

Linearity: Example 4

Consider the system $y(t) = 3x(t) + 2$

Additive? No:

$$S\Big\{x_1(t) + x_2(t)\Big\} = 3\Big(x_1(t) + x_2(t)\Big) + 2$$

On the other hand,

$$S\Big\{x_1(t)\Big\} + S\Big\{x_2(t)\Big\} = 3\Big(x_1(t) + 3x_2\Big) + 4$$

$$S\Big\{x_1(t) + x_2(t)\Big\} \neq S\Big\{x_1(t)\Big\} + S\Big\{x_2(t)\Big\}$$

Homogeneous? No:

$$S\Big\{ax(t)\Big\} = 3ax(t) + 2$$

On the other hand,

$$aS\Big\{x(t)\Big\} = 3ax(t) + 2a \neq 3ax(t) + 2 \qquad \text{unless } a = 1$$

This system is not linear.

Time invariance

A system S is **time-invariant** if, for any input $x(t)$ and any fixed time t_1, the output $S\Big\{x(t-t_1)\Big\}$ is equal to $y(t-t_1)$, where $y(t)$ is the output due to $x(t)$, i.e., $y(t) = S\Big\{x(t)\Big\}$.

Systems that are not time-invariant are called **time-varying**.

Classic example: systems described by linear differential equations with constant coefficients, such as

$$5\frac{d^2y(t)}{dt^2} - 3y(t) = -\frac{dx(t)}{dt} + 2x(t).$$

Linear (RLC) circuits are described in this way.

Time invariance: Example 1

Consider the system

$$y(t) = 3x^2(t)u(t)$$

We have

$$S\Big\{x(t-t_1)\Big\} = 3x^2(t-t_1)u(t).$$

On the other hand,

$$\begin{aligned} y(t-t_1) &= 3x^2(t-t_1)u(t-t_1) \\ &\neq 3x^2(t-t_1)u(t) \quad \text{unless } t_1 = 0 \end{aligned}$$

This system is time-varying.

Time invariance: Example 2

Consider the system

$$y(t) = \int_0^t e^{-2\tau} x(\tau) d\tau$$

We have

$$S\Big\{x(t)\Big\} = \int_0^t e^{-2\tau} x(\tau) d\tau.$$

Then

$$S\Big\{x(t-t_1)\Big\} = \int_0^t e^{-2\tau} x(\tau - t_1) d\tau = e^{-2t_1} \int_{-t_1}^{t-t_1} e^{-2\tau} x(\tau) d\tau$$

and $\quad y(t-t_1) = \int_0^{t-t_1} e^{-2\tau} x(\tau) d\tau.$

Since $S\Big\{x(t-t_1)\Big\} \neq y(t-t_1)$, this system is time-varying.

Time invariance: Example 3

Consider the system

$$\frac{d^2 y(t)}{dt^2} = -3x(t)$$

In other words, if $y(t) = S\Big\{x(t)\Big\}$, then

$$\frac{d^2 y(t)}{dt} = -3x(t).$$

Let $v(t) = x(t - t_1)$. So if $z(t) = S\Big\{v(t)\}$, then

$$\frac{d^2 z(t)}{dt^2} = -3v(t) = -3x(t - t_1) = \frac{d^2 \big(y(t - t_1)\big)}{dt^2}.$$

This system is time-invariant.

Linear time-invariant (LTI) systems

We will focus almost exclusively on **linear time-invariant** (LTI) systems. We will prove later that any such system has a **convolution representation**

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau \qquad \text{(continuous-time)}$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[n-k]x[k] \qquad \text{(discrete-time)}$$

where h is called the impulse response of the system.

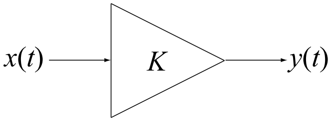
Another important property of LTI systems is their action on complex exponentials: if S is LTI, then

$$S\left\{e^{j\omega t}\right\} = c(\omega)e^{j\omega t}$$

for some complex number $c(\omega)$. So LTI systems can attenuate or amplify various frequency components of the input.

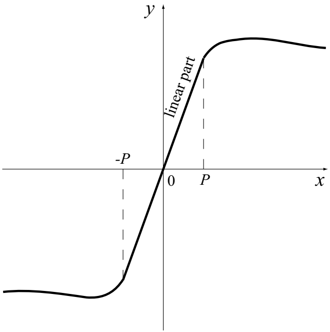
An **ideal amplifier**

$$y(t) = Kx(t), \quad K > 0$$



is linear.

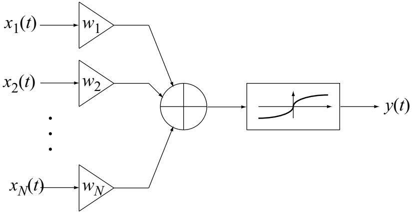
However, real amplifiers have saturation effects:



$y(t) = Kx(t)$ in the **linear range** $-P \le x(t) \le P$, but the output **saturates** at some value $\approx \pm KP$ for $|x(t)| \gg P$.

Nonlinear systems

The **summing neuron** model used in artificial neural networks and in mathematical models of biological neural systems:



$$y(t) = \sigma\left(\sum_{n=1}^{N} w_n x_n(t)\right),$$

where:

- $x_1(t), \ldots, x_N(t)$ are N inputs
- $w_1, \ldots, w_N$ are the synaptic weights
- $\sigma(\cdot)$ is a nonlinear transformation

This is an example of a **multiple-input, single-output** system.