Lecture II: Continuous-Time and Discrete-Time Signals

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BME 171: Signals and Systems
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This lecture

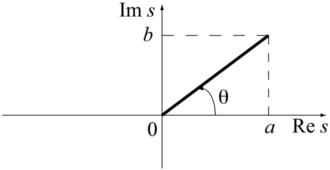
Plan for the lecture:

1. Review: complex numbers
2. Continuous-time signals
   - unit step and unit ramp
   - unit impulse
   - transformations of time
3. Discrete-time signals
   - unit step
   - unit impulse
4. Periodic continuous-time and discrete-time signals

Review: complex numbers

- **Rectangular form:** $s = a + jb, \qquad j = \sqrt{-1}$
  $a = \text{Re}(s), \quad b = \text{Im}(s)$
- **Polar form:** $s = re^{j\theta}$
- **Euler's formula:** $e^{j\theta} = \cos\theta + j\sin\theta$
- **Complex conjugate:** $s^* = a - jb = re^{-j\theta}$

$$ss^* = |s|^2 = a^2 + b^2$$



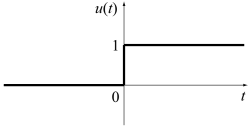
$$r = |s| = \sqrt{a^2 + b^2}$$

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

Unit step and unit ramp

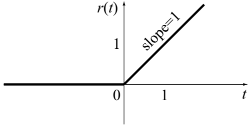
Unit step:

$$u(t) = \begin{cases} 1, & t \ge 0 \\ 0, & t < 0 \end{cases}$$



Unit ramp:

$$r(t) = \begin{cases} t, & t \ge 0 \\ 0, & t < 0 \end{cases}$$



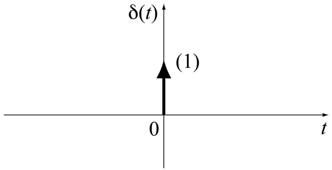
Running integral representation:

$$r(t) = \int_{-\infty}^{t} u(\tau) d\tau$$

Unit impulse

Unit impulse (aka Dirac delta-function):

1. $\delta(t) = 0$ for $t \neq 0$
2. $\int_{-a}^{a} \delta(t)dt = 1$ for any $a > 0$



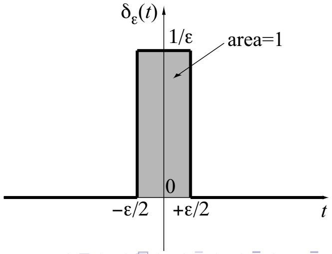
The value of $\delta(t)$ at $t = 0$ is undefined; in particular, it is not $+\infty$!

It is useful to think of $\delta(t)$ as an infinitesimally narrow pulse of unit area centered around 0:

$$\delta(t) = \lim_{\varepsilon \to 0} \delta_\varepsilon(t),$$

where

$$\delta_\varepsilon(t) = \begin{cases} 1/\varepsilon, & -\varepsilon/2 \le t \le \varepsilon/2 \\ 0, & |t| > \varepsilon/2 \end{cases}$$



The main property of the unit impulse

If $x(t)$ is a signal that is continuous at $t = 0$, then

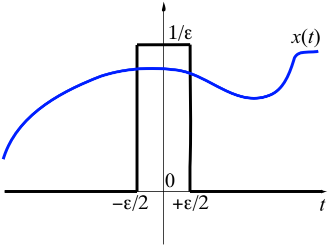
$$x(t)\delta(t) = x(0)\delta(t).$$

In particular,

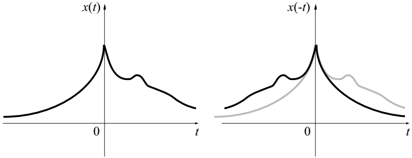
$$\int_{-a}^{a} x(t)\delta(t)dt = x(0) \qquad \text{for any } 0 < a \le +\infty.$$

You can convince yourselves of this by approximating $\delta(t)$ with a pulse, such as $\delta_\varepsilon(t)$, and using the fact that, if ε is small enough, then

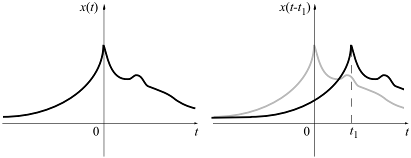
$$x(t) \approx x(0) \qquad \text{for } -\varepsilon/2 \le t \le \varepsilon/2.$$



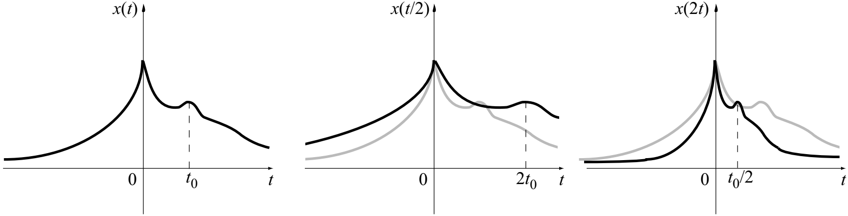
Time reversal: $x(t) \longrightarrow x(-t)$



Time shifts: $x(t) \longrightarrow x(t - t_1)$

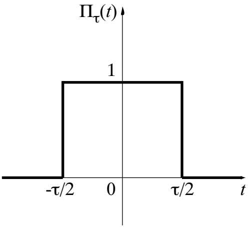


Time scaling: $x(t) \longrightarrow x(ct)$

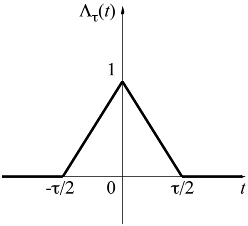


Examples

$$\Pi_\tau(t) = u\left(t + \frac{\tau}{2}\right) - u\left(t - \frac{\tau}{2}\right)$$

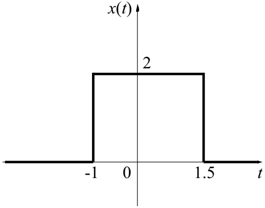


$$\Lambda_\tau(t) = \frac{2}{\tau}\left(t + \frac{\tau}{2}\right)\Pi_{\tau/2}\left(t + \frac{\tau}{4}\right) - \frac{2}{\tau}\left(t - \frac{\tau}{2}\right)\Pi_{\tau/2}\left(t - \frac{\tau}{4}\right)$$

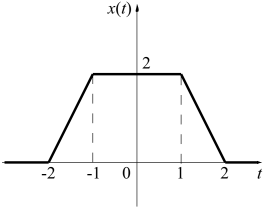


More examples

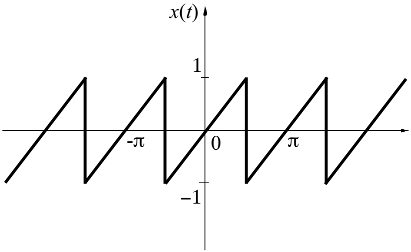
$$x(t) = 2\Pi_{2.5}(t - 0.25)$$



$$x(t) = 2(t+2)\Pi_1(t+1.5) - 2(t-2)\Pi_1(t-1.5) + 2\Pi_2(t)$$

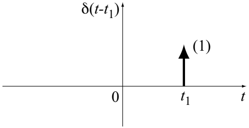


$x(t) = \sum_{k=-\infty}^{\infty} g(t - k\pi)$, where $g(t) = t\Pi_\pi(t)$

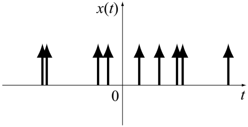


Shifted unit impulse and the sifting property

Unit impulse located at $t = t_1$:



Example: neural spike trains



$$x(t) = \sum_{k=1}^{K} \delta(t - t_k)$$

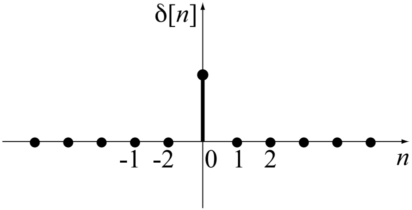
t_k, $1 \le k \le K$: spike times
interspike intervals $t_{k+1} - t_k$: milliseconds

The **sifting property** of the unit impulse: for any signal $x(t)$ that's continuous at $t = t_1$,

$$\int_{-\infty}^{\infty} x(t)\delta(t - t_1)dt = x(t_1)$$

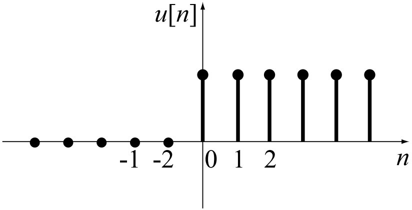
Discrete-time unit impulse:

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n = \pm 1, \pm 2, \dots \end{cases}$$



Discrete-time unit step:

$$u[n] = \begin{cases} 1, & n = 0, 1, 2, \dots \\ 0, & n = -1, -2, \dots \end{cases}$$



It is easy to see that

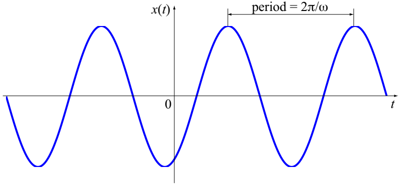
$$x[n]\delta[n] = \begin{cases} x[0], & n = 0 \\ 0, & n = \pm 1, \pm 2, \dots \end{cases} \qquad u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$

$x(t)$ is **periodic** if there exists a number $T > 0$, such that

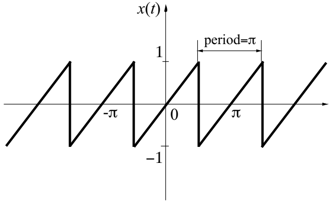
$$x(t+T) = x(t), \qquad \text{for all } t.$$

Fundamental period: smallest positive T, such that the above holds.

Examples:



sinusoid $x(t) = A\cos(\omega t + \theta)$



triangular wave

Sums of periodic signals

Suppose $x_1(t)$ is periodic with period T_1 and $x_2(t)$ is periodic with period T_2. Then

$$x(t) = x_1(t) + x_2(t)$$

is periodic if and only if there exist positive integers q and r, such that $rT_1 = qT_2$. Moreover, if r and q are relatively prime (i.e., have no common multiple except 1), then $T = rT_1$ is the fundamental period of $x(t)$.

Example:

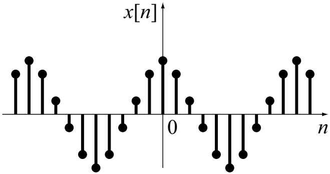
- $x(t) = 5\cos(3\pi t + 1.2) - 8\sin(5\pi t - 4)$ is periodic
- $x(t) = 5\cos(3\pi t + 1.2) - 8\sin(5t - 4)$ is not periodic

Periodic discrete-time signals

$x[n]$ is **periodic** if there exists a positive **integer** T, such that

$$x[n+T] = x[n], \qquad \text{for all } n.$$

Fundamental period: smallest positive integer T, such that the above holds.



fundamental period = 6

Example: $x[n] = A\cos(\Omega n + \theta)$ is periodic if and only if there are positive integers q and r, such that $\Omega = 2\pi q/r$ (in other words, if Ω is a **rational** multiple of 2π).