Lecture XVI: Solving LDEs via the Laplace transform

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This lecture

Plan for the lecture:

1. Causal LTI systems described by differential equations
2. First-order systems
    - example: RC lowpass flter
3. Second-order systems
4. Nth-order systems

Causal LTI systems described by differential equations

We will now see how to use the Laplace transform to analyze the behavior of causal LTI systems that are described by differential equations.

We will consider input-output equations of the form

$$\frac{d^N y(t)}{dt^N} + \sum_{i=0}^{N-1} a_i \frac{d^i y(t)}{dt^i} = \sum_{i=0}^{M} b_i \frac{d^i x(t)}{dt^i}$$

Here, $x(t)$ is the input signal and $y(t)$ is the output signal. We assume that $x^{(i)}(0^-) = 0$ for all $i = 1, \ldots, M-1$. We will also assume that $M \leq N$.

We assume that the initial conditions (ICs) are specified via the values

$$y(0^-), y'(0^-), \ldots, y^{(N-1)}(0^-).$$

First-order systems

Consider first a first-order system

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

Let us take the Laplace transform of both sides to get

$$sY(s) - y(0^-) + aY(s) = bX(s)$$

Solving for $Y(s)$, we get the s-**domain representation of the system:**

$$Y(s) = \frac{y(0^-)}{s+a} + \frac{b}{s+a}X(s)$$

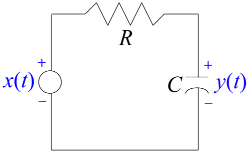
Here, the first term on the right-hande side is the **zero-input response** (ZIR) of the system, while the second term is the **zero-state response** (ZSR). If $y(0^-) = 0$, then we have

$$Y(s) = H(s)X(s), \qquad \text{where } H(s) = \frac{b}{s+a}.$$

We call $H(s)$ the **transfer function** of the system.

Example: RC lowpass filter

Consider the RC lowpass filter:



$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$$

Then $a = b = 1/RC$, and we can write the s-domain input-output relation as

$$Y(s) = \frac{y(0^-)}{s + 1/RC} + \frac{1/RC}{s + 1/RC}X(s)$$

Consider the input $x(t) = u(t)$, so that $X(s) = 1/s$. Then, using partial fraction expansions, we get

$$Y(s) = \frac{y(0^-)}{s + 1/RC} + \frac{1/RC}{s + 1/RC}\frac{1}{s} = \frac{y(0^-)}{s + 1/RC} + \frac{1}{s} - \frac{1}{s + 1/RC}.$$

We can compute the inverse Laplace transform to get

$$y(t) = \left(y(0^-)e^{-t/RC} + 1 - e^{-t/RC}\right)u(t)$$

Second-order systems

Now consider

$$\frac{d^2y(t)}{dt^2} + a_1\frac{dy(t)}{dt} + a_0y(t) = b_1\frac{dx(t)}{dt} + b_0x(t)$$

Take the Laplace transform of both sides:

$$s^2Y(s) - sy(0^-) - y'(0^-) + a_1\left[sY(s) - y(0^-)\right] + a_0Y(s) = b_1sX(s) + b_0X(s)$$

(we used our standing assumption that $x(0^-) = 0$). Solve for $Y(s)$:

$$Y(s) = \frac{y(0^-)s + y'(0^-) + a_1y(0^-)}{s^2 + a_1s + a_0} + \frac{b_1s + b_0}{s^2 + a_1s + a_0}X(s)$$

As before, the first term on the right-hand side is the ZIR, while the second term is the ZSR. If $y(0^-) = y'(0^-) = 0$, then we have

$$Y(s) = H(s)X(s), \qquad \text{where } H(s) = \frac{b_1s + b_0}{s^2 + a_1s + a_0}.$$

Note that the transfer function of a 2nd-order system is a 2nd-order rational function.

Nth-order system

The same method can be used to derive the s-domain representation of an Nth-order system

$$\frac{d^N y(t)}{dt^N} + \sum_{i=0}^{N-1} a_i \frac{d^i y(t)}{dt^i} = \sum_{i=0}^{M} b_i \frac{d^i x(t)}{dt^i}$$

We will get

$$Y(s) = \frac{C(s)}{A(s)} + \frac{B(s)}{A(s)} X(s),$$

where $C(s)$ is determined by the ICs $y^{(i)}(0^-), i = 0, 1, \ldots, N-1$, and

$$\begin{aligned} B(s) &= b_M s^M + \ldots + b_1 s + b_0 \\ A(s) &= s^N + a_{N-1} s^{N-1} + \ldots + a_1 s + a_0. \end{aligned}$$

Observe that the poles of the transfer function $H(s) = B(s)/A(s)$ are determined by the coefficients multiplying the output signal and its derivatives, while the zeros of $H(s)$ are determined by the coefficients multiplying the input signal and its derivatives.