Lecture XV: Inverse Laplace transform

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This lecture

Plan for the lecture:

1. Recap: the one-sided Laplace transform
2. Inverse Laplace transform: the Bromwich integral
3. Inverse Laplace transform of a rational function
    - poles, zeros, order
4. Partial fraction expansions
    - Distinct poles
    - Repeated poles
    - Improper rational functions
    - Transforms containing exponentials
5. Pole locations and the form of a signal

Recap: the (one-sided) Laplace transform

Given a causal signal $x(t)$ (i.e., $x(t) = 0$ for $t < 0$), we have defined its *one-sided* Laplace transform as

$$X(s) \triangleq \int_0^\infty x(t)e^{-st}dt$$

The Laplace transform is a powerful tool for solving differential equations, finding the response of an LTI system to a given input and for stability analysis.

The inverse Laplace transform

We can also define the **inverse Laplace transform**: given a function $X(s)$ in the s-domain, its inverse Laplace transform $\mathcal{L}^{-1}[X(s)]$ is a function $x(t)$ such that $X(s) = \mathcal{L}[x(t)]$. It can be shown that the Laplace transform of a causal signal is unique; hence, the inverse Laplace transform is uniquely defined as well.

In general, the computation of inverse Laplace transforms requires techniques from complex analysis. The simplest inversion formula is given by the so-called **Bromwich integral**

$$x(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} X(s) e^{st} ds,$$

where the integral is evaluated along the path from $s = c - j\infty$ to $s = c + j\infty$ for any real c such that this path lies in the ROC.

Inverse Laplace transform of rational functions

However, for a wide class of functions the inverse Laplace transform can be computed using algebraic techniques. These are the so-called **rational functions**, or ratios of polynomials in s.

Suppose that the Laplace transform of some signal $x(t)$ has the form

$$X(s) = \frac{B(s)}{A(s)},$$

where $B(s)$ and $A(s)$ are polynomials in the complex variable s:

$$\begin{aligned} B(s) &= b_M s^M + b_{M-1} s^{M-1} + \ldots + b_1 s + b_0 \\ A(s) &= a_N s^N + a_{N-1} s^{N-1} + \ldots + a_1 s + a_0 \end{aligned}$$

Here, M and N are positive integers and the coefficients $b_M, b_{M-1}, \ldots, b_0$ and $a_N, a_{N-1}, \ldots, a_0$ are real numbers. We assume that $b_M, a_N \neq 0$. We assume that $B(s)$ and $A(s)$ have no common factors.

Rational functions

$$X(s) = \frac{B(s)}{A(s)} = \frac{b_M s^M + b_{M-1} s^{M-1} + \ldots + b_1 s + b_0}{a_N s^N + a_{N-1} s^{N-1} + \ldots + a_1 s + a_0}$$

Some terminology:

- $B(s)$: the numerator polynomial
- $A(s)$: the denominator polynomial
- N (the degree of the denominator polynomial): the order of the rational function
- Let $p_1, \ldots, p_N$ be the roots of $A(s)$, i.e.,

$$A(s) = a_N (s - p_1)(s - p_2) \ldots (s - p_N)$$

  Then we call $p_1, \ldots, p_N$ the poles of $X(s)$.

- Let $z_1, \ldots, z_M$ be the roots of $B(s)$, i.e.,

$$B(s) = b_M (s - z_1)(s - z_2) \ldots (s - z_M)$$

  Then we call $z_1, \ldots, z_M$ the zeros of $X(s)$.

Inverse Laplace transforms via partial fraction expansions

Let us write $X(s)$ in the form

$$X(s) = \frac{B(s)}{a_N(s-p_1)(s-p_2)\dots(s-p_N)}$$

We will use this factorization to decompose $X(s)$ into *partial fractions* and then use known Laplace transform pairs to compute the inverse Laplace transform $\mathcal{L}^{-1}[X(s)]$.

We assume for now that the rational function $X(s)$ is **proper**, i.e., $M < N$. We will consider the opposite case later.

Depending on the structure of the set of poles of $X(s)$, the methodology for finding the partial fraction expansion will differ. We will consider several cases.

Distinct poles, all real

We first consider the case when all the poles $p_1, \ldots, p_N$ are real and distinct (i.e.,$p_i \neq p_j$ when $i \neq j$). Then $X(s)$ has the partial fraction expansion

$$X(s) = \frac{c_1}{s - p_1} + \frac{c_2}{s - p_2} + \ldots + \frac{c_N}{s - p_N}$$

where the constants $c_1, \ldots, c_N$ (called the **residues** of $X(s)$) are determined via the formula

$$c_i = \left[(s - p_i)X(s)\right]_{s=p_i}, \qquad i = 1, \ldots, N$$

We can now compute $\mathcal{L}^{-1}[X(s)]$ using linearity and the transform pair

$$e^{-bt}u(t) \leftrightarrow \frac{1}{s+b}.$$

Thus,

$$x(t) = \mathcal{L}^{-1}[X(s)] = (c_1 e^{p_1 t} + c_2 e^{p_2 t} + \ldots + c_N e^{p_N(t)})u(t)$$

Example

$$X(s) = \frac{s+1}{s^2+7s+12}$$

We have $B(s) = s+1$ and $A(s) = s^2+7s+12$. We can factor $A(s)$ as

$$A(s) = (s+4)(s+3),$$

so the poles are

$$p_1 = -4 \qquad \text{and} \qquad p_2 = -3.$$

Compute the residues:

$$c_1 = [(s+4)X(s)]_{s=-4} = \left[\frac{s+1}{s+3}\right]_{s=-4} = 3$$

$$c_2 = [(s+3)X(s)]_{s=-3} = \left[\frac{s+1}{s+4}\right]_{s=-3} = -2.$$

Hence,

$$x(t) = (3e^{-4t} - 2e^{-3t})u(t)$$

Distinct poles, two or more complex

Now suppose that the poles are still all distinct, but two or more of them are complex. Since complex roots of polynomials occur in conjugate pairs, for each complex pole p there will be another complex pole $\bar{p}$.

Suppose that $p_1 = \sigma + j\omega$ is a complex pole; let us reorder the poles so that $p_2 = \bar{p}_1 = \sigma - j\omega$. Let us now assume that the rest of the poles $(p_3, \ldots, p_N)$ are real. Then we have

$$c_1 = \left[(s - p_1)X(s)\right]_{s=p_1}$$

and

$$c_2 = \left[(s - p_2)X(s)\right]_{s=p_2} = \left[(s - \bar{p}_1)X(s)\right]_{s=\bar{p}_1} = \bar{c}_2.$$

Hence, the partial fraction expansion will be

$$X(s) = \frac{c_1}{s - p_1} + \frac{\bar{c}_1}{s - \bar{p}_1} + \frac{c_3}{s - p_3} + \ldots + \frac{c_N}{s - p_N}$$

Distinct poles, two or more complex: cont'd

Inverting, we get

$$x(t) = c_1 e^{p_1 t} + \bar{c}_1 e^{\bar{p}_1 t} + c_3 e^{p_3 t} + \ldots + c_N e^{p_N t}, \qquad t \ge 0$$

It can be verified (exercise!) that the first two terms can be combined as follows:

$$c_1 e^{p_1 t} + \bar{c}_1 e^{\bar{p}_1} = 2|c_1| e^{\sigma t} \cos(\omega t + \angle c_1),$$

where $|c_1|$ and $\angle c_1$ are the magnitude and the phase of the residue c_1, and σ and ω are the real and the imaginary parts of the pole p_1.

Thus, we can write

$$x(t) = \left[2|c_1| e^{\sigma t} \cos(\omega t + \angle c_1) + c_3 e^{p_3 t} + \ldots + c_N e^{p_N t} \right] u(t)$$

Example

$$X(s) = \frac{-s^2+1}{s^3+9s}$$

$B(s) = -s^2+1,\ A(s) = s^3+9s = s(s^2+9) = s(s+3j)(s-3j)$.
The poles are

$$p_1 = 3j, p_2 = -3j, p_3 = 0$$

Compute the residues:

$$c_1 = [(s-3j)X(s)]_{s=3j} = \left[\frac{-s^2+1}{s(s+3j)}\right]_{s=3j} = \frac{10}{3j\cdot 6j} = -\frac{5}{9}$$

$$c_2 = \bar{c}_1 = -\frac{5}{9}; \qquad c_1 = \frac{5}{9}e^{j\pi}$$

$$c_3 = [sX(s)]_{s=0} = \left[\frac{-s^2+1}{(s-3j)(s+3j)}\right]_{s=0} = \frac{1}{-3j\cdot 3j} = \frac{1}{9}$$

Thus,

$$x(t) = \left[\frac{10}{9}\cos(3t+\pi) + \frac{1}{9}\right]u(t)$$

Repeated poles

Now let us consider the case when the pole p_1 of $X(s)$ is repeated r times, while the other $N-r$ poles $p_{r+1}, p_{r+2}, \ldots, p_N$ are distinct. Thus, we have reordered the poles so that $p_1 = p_2 = \ldots = p_r$. Then the partial fraction expansion will be

$$X(s) = \frac{c_1}{s-p_1} + \frac{c_2}{(s-p_1)^2} + \ldots + \frac{c_r}{(s-p_1)^r} + \frac{c_{r+1}}{s-p_{r+1}} + \ldots + \frac{c_N}{s-p_N}$$

The residues $c_{r+1}, \ldots, c_N$ are computed as before:

$$c_i = [(s-p_i)X(s)]_{s=p_i}, \qquad i = r+1, \ldots, N;$$

the rth residue c_r is given by

$$c_r = [(s-p_1)^r X(s)]_{s=p_1}$$

and the residues $c_1, c_2, \ldots, c_{r-1}$ are given by

$$c_{r-i} = \frac{1}{i!}\left[\frac{d^i}{ds^i}\left[(s-p_1)^r X(s)\right]\right]_{s=p_1}, \qquad i = 1, 2, \ldots, r-1$$

Repeated poles: cont'd

If the poles of $X(s)$ are all real, then we can compute the inverse Laplace transform using the transform pairs

$$\frac{t^{N-1}}{(N-1)!}e^{-at} \leftrightarrow \frac{1}{(s+a)^N}, \qquad N = 1, 2, 3, \dots$$

Thus the inverse Laplace transform of

$$X(s) = \frac{c_1}{s-p_1} + \frac{c_2}{(s-p_1)^2} + \dots + \frac{c_r}{(s-p_1)^r} + \frac{c_{r+1}}{s-p_{r+1}} + \dots + \frac{c_N}{s-p_N}$$

is given by

$$x(t) = \Big(c_1 e^{p_1 t} + c_2 t e^{p_1 t} + \dots + \frac{c_r t^{r-1}}{(r-1)!}e^{p_1 t} + c_{r+1}e^{p_{r+1}t} + \dots + c_N e^{p_N t}\Big)u(t)$$

If $X(s)$ has complex repeated poles, the complex part of $X(s)$ can be expanded in terms of powers of quadratic terms.

Example

$$X(s) = \frac{s-1}{s^3 + 2s^2 + s}$$

We have

$$X(s) = \frac{s-1}{s(s+1)^2},$$

so the poles are

$$p_1 = p_2 = -1, p_3 = 0$$

Compute the residues:

$$c_1 = \left[\frac{d}{ds}\left[(s+1)^2 X(s)\right]\right]_{s=-1} = \left[\frac{d}{ds}\left(1 - \frac{1}{s}\right)\right]_{s=-1} = \left[\frac{1}{s^2}\right]_{s=-1} = 1$$

$$c_2 = \left[(s+1)^2 X(s)\right]_{s=-1} = \left[1 - \frac{1}{s}\right]_{s=-1} = 2$$

$$c_3 = \left[sX(s)\right]_{s=0} = \left[\frac{s-1}{(s+1)^2}\right]_{s=0} = -1$$

Thus,

$$x(t) = \left(e^{-t} + 2te^{-t} - 1\right) u(t)$$

Improper rational functions

Consider the case

$$X(s) = \frac{B(s)}{A(s)},$$

where $B(s)$ and $A(s)$ are polynomials in s of degrees M and N, respectively, but $M \geq N$. Then we can first write $X(s)$ in the form

$$X(s) = Q(s) + \frac{R(s)}{A(s)}$$

using long division of polynomials (or a symbolic manipulation package). Then the quotient $Q(s)$ can be handled using the transform pair

$$\frac{d^N}{dt^N}\delta(t) \leftrightarrow t^N, \qquad N = 1, 2, \ldots$$

while the term $Q(s)/A(s)$, which is now proper, can be handled using partial fraction expansions.

Transforms containing exponentials

We can also compute inverse Laplace transforms of functions that can be written in the form

$$X(s) = \frac{B_0(s)}{A_0(s)} + \sum_{i=1}^{q} \frac{B_i(s)}{A_i(s)} e^{-h_i s},$$

where $h_1, \ldots, h_q > 0$. First, we use partial fraction expansions to compute the inverse Laplace transforms

$$x_i(t) = \mathcal{L}^{-1}\left[\frac{B_i(s)}{A_i(s)}\right], \qquad i = 0, 1, \ldots, q.$$

Then use the time-shift property

$$\mathcal{L}[x(t-c)u(t-c)] = e^{-cs}X(s), \qquad c > 0$$

to write

$$\mathcal{L}^{-1}\left[\frac{B_i(s)}{A_i(s)} e^{-h_i s}\right] = x_i(t - h_i), \qquad i = 1, \ldots, q.$$

Finally, use linearity to get $x(t) = x_0(t) + \sum_{i=1}^{q} x_i(t - h_i)u(t - h_i)$

Pole locations and the form of a signal

When the signal $x(t)$ has a rational Laplace transform $X(s)$, we can tell a great deal about the functional form of $x(t)$ without inverting $X(s)$, simply by looking at the poles. To this end, we can use the following handy table:

If $X(s)$ has:	**Then $x(t)$ contains a term ($\times u(t)$):**
a nonrepeated real pole p	ce^{pt}
a real pole p repeated twice	$c_1 e^{pt} + c_2 t e^{pt}$
a single pair of complex poles $\sigma \pm j\omega$	$ce^{\sigma t} \cos(\omega t + \theta)$
a pair of poles $\sigma \pm j\omega$ repeated twice	$c_1 e^{\sigma t} \cos(\omega t + \theta_1) + c_2 t e^{\sigma t} \cos(\omega t + \theta_2)$

Here, the constants $c, c_1, c_2, \theta_1, \theta_2$ are determined by the zeros of $X(s)$.