Lecture XIV: The Laplace transform

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This lecture

Plan for the lecture:

1. Recap: the Fourier transform
2. Motivation for the Laplace transform: exponential decaying factors
3. The Laplace transform: definition
4. Properties of the Laplace transform

Recap: The Fourier transform

We have seen earlier that we can represent sufficiently regular (e.g., absolutely integrable) signals via their Fourier transform:

$$\begin{aligned} x(t) &\leftrightarrow X(\omega) \\ X(\omega) &= \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \end{aligned}$$

However, many important types of signals do not have a Fourier transform in the ordinary sense, i.e., the integral that defines $X(\omega)$ diverges.

Ex.: $x(t) = u(t)$

$$\int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{0}^{\infty} e^{-j\omega t} dt = \lim_{T \to \infty} \left[-\frac{1}{j\omega} e^{-j\omega t} \right]_0^T,$$

and this limit does not exist because of the oscillatory behavior of $e^{-j\omega t}$.

A possible way out

We can remedy this situation by introducing an exponentially decaying term $e^{-\sigma t}$, where σ is a real constant.

Again, consider $x(t) = u(t)$.

$$\begin{aligned} \int_{-\infty}^{\infty} u(t) e^{-\sigma t} e^{-j\omega t} dt &= \int_{0}^{\infty} e^{-(\sigma+j\omega)t} dt \\ &= \lim_{T\to\infty} \left[-\frac{1}{\sigma + j\omega} e^{-(\sigma+j\omega)t} \right]_0^T \\ &= \frac{1}{\sigma + j\omega} - \frac{1}{\sigma + j\omega} \lim_{T\to\infty} \left[e^{-\sigma T} e^{-j\omega T} \right]. \end{aligned}$$

If $\sigma > 0$, then the above limit is equal to 0; if $\sigma \le 0$, the limit does not exist. Thus, if $\sigma > 0$, we have

$$\int_{-\infty}^{\infty} u(t) e^{-(\sigma+j\omega)t} dt = \frac{1}{\sigma + j\omega}.$$

The Laplace transform

This leads us to the definition of the **Laplace transform**. Given a signal $x(t)$ and a complex number $s = \sigma + j\omega$, we define

$$X(s) \triangleq \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt.$$

The above integral may diverge for some values of s. The set of all complex numbers s for which the above integral converges is called the **region of convergence (ROC)** of the Laplace transform of $x(t)$.

This is called the **bilateral** (or **two-sided**) Laplace transform. We can also define the **unilateral** (or **one-sided**) Laplace transform:

$$X(s) = \int_{0}^{\infty} x(t)e^{-st}dt.$$

We will deal with the one-sided Laplace transform, because that will allow us to deal conveniently with systems that have nonzero initial conditions.

The Laplace transform: an example

As an example, let us consider $x(t) = e^{-bt}u(t)$, where b is real or complex.

$$\begin{aligned} X(s) &= \int_0^\infty x(t)e^{-st}dt \\ &= \int_0^\infty e^{-(b+s)t}dt \\ &= \lim_{T\to\infty}\left[-\frac{1}{b+s}e^{-(b+s)t}\right]_0^T \\ &= \frac{1}{b+s} - \frac{1}{b+s}\lim_{T\to\infty}\left[e^{-(b+s)T}\right]. \end{aligned}$$

The above limit will be equal to 0 if $\mathrm{Re}(b+s) > 0$; otherwise, it does not exist. Hence,

$$X(s) = \frac{1}{s+b}, \qquad \text{if } \mathrm{Re}(b+s) > 0.$$

Thus, the ROC is the set of all complex numbers s, such that $\mathrm{Re}\, s > -\mathrm{Re}\, b$.

The relation to the Fourier transform

A word of caution: it is tempting to assume that we can compute the Fourier transform of $x(t)$ from its Laplace transform $X(s)$ by setting $s = j\omega$ (i.e., $\sigma = 0$). However, this is valid if and only if the set $\{s \in \mathbb{C} : \mathrm{Re}\, s = 0\}$ is in the ROC of $X(s)$.

Ex.: $x(t) = e^{-bt}u(t)$, b real.

The ROC is $\{s \in \mathbb{C} : \mathrm{Re}\, s > -b\}$. If $b > 0$, then the ROC includes all purely imaginary numbers. Thus,

$$X(\omega) = \left.\frac{1}{s+b}\right|_{s=j\omega} = \frac{1}{j\omega + b},$$

which we have proved before.

On the other hand, if $b \le 0$, the ROC does not contain any purely imaginary number, so the Fourier transform does not exist, while the Laplace transform is equal to

$$X(s) = \frac{1}{s+b}$$

for any s in the ROC.

Properties of the Laplace transform

Just as the Fourier transform, the Laplace transform has a number of useful properties which we will learn to exploit. We will look at and prove a few of these.

We will use the notation $x(t) \leftrightarrow X(s)$ to denote the fact that $X(s)$ is the Laplace transform of $x(t)$; we will also use the notation $X(s) = \mathcal{L}[x(t)]$. First of all, the Laplace transform is linear: if $X_1(s) = \mathcal{L}[x_1(t)]$ and $X_2(s) = \mathcal{L}[x_2(t)]$, then for any constants a_1 and a_2

$$\mathcal{L}[a_1x_1(t) + a_2x_2(t)] = a_1\mathcal{L}[x_1(t)] + a_2\mathcal{L}[x_2(t)] = a_1X_1(s) + a_2X_2(s).$$

This is obvious from definitions.

IF $X(s) = \mathcal{L}[x(t)]$, then for any real number $c > 0$ we have

$$x(t-c)u(t-c) \leftrightarrow e^{-cs}X(s).$$

Proof:

$$\begin{aligned}
\mathcal{L}[x(t-c)u(t-c)] &= \int_0^\infty x(t-c)u(t-c)e^{-st}dt \\
&= \int_c^\infty x(t-c)e^{-st}dt \\
&= \int_0^\infty x(t)e^{-s(t+c)}dt \\
&= e^{-cs}\int_0^\infty x(t)e^{-st}dt \\
&= e^{-cs}X(s).
\end{aligned}$$

Q.E.D.

Right shift in time: example

Ex.: consider the rectangular pulse

$$x(t) = \begin{cases} 1, & 0 \le t < c \\ 0, & \text{otherwise} \end{cases}$$

We can write

$$x(t) = u(t) - u(t-c).$$

Then, using linearity and shift property we get

$$X(s) = \frac{1}{s} - \frac{e^{-cs}}{s} = \frac{1 - e^{-cs}}{s}.$$

The ROC is $\{s \in \mathbb{C} : \mathrm{Re}\, s > 0\}$.

Properties: multiplication by a power of t

If $X(s) = \mathcal{L}[x(t)]$, then

$$\mathcal{L}[t^N x(t)] = (-1)^N \frac{d^N}{ds^N} X(s)$$

Proof: similar to the corresponding property of the Fourier transform.

Ex.: $x(t) = t^N u(t)$.

$$X(s) = (-1)^N \frac{d^N}{ds^N}\left(\frac{1}{s}\right) = \frac{N!}{s^{N+1}}.$$

In particular, for the unit ramp $x(t) = tu(t)$ we have

$$X(s) = \frac{1}{s^2}.$$

Ex.: $x(t) = te^{-bt}u(t)$.

$$X(s) = (-1)\frac{d}{ds}\left(\frac{1}{s+b}\right) = \frac{1}{(s+b)^2}.$$

Properties: multiplication by an exponential

If $X(s) = \mathcal{L}[x(t)]$, then

$$\mathcal{L}[e^{at}x(t)] = X(s-a)$$

for any a, real or complex.

Proof:

$$\begin{aligned}
\mathcal{L}[e^{at}x(t)] &= \int_0^\infty x(t)e^{at}e^{-st}dt \\
&= \int_0^\infty x(t)e^{-(s-a)t}dt \\
&= X(s-a).
\end{aligned}$$

Ex.: $x(t) = [u(t) - u(t-c)]e^{at}$, where $c > 0$ and a is a real number. Then

$$X(s) = \frac{1 - e^{-c(s-a)}}{s-a}.$$

Laplace transform of cosine and sine

1. $x(t) = \cos(\omega_0 t)u(t) = \frac{1}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]u(t)$

$$\begin{aligned} X(s) &= \frac{1}{2}\left[\frac{1}{s-j\omega_0} + \frac{1}{s+j\omega_0}\right] \\ &= \frac{1}{2}\cdot\frac{2s}{s^2+\omega_0^2} \\ &= \frac{s}{s^2+\omega_0^2}. \end{aligned}$$

2. $x(t) = \sin(\omega_0 t)u(t) = \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right]$

$$\begin{aligned} X(s) &= \frac{1}{2j}\left[\frac{1}{s-j\omega_0} - \frac{1}{s+j\omega_0}\right] \\ &= \frac{1}{2j}\cdot\frac{2j\omega_0}{s^2+\omega^2+0} \\ &= \frac{\omega_0}{s^2+\omega_0^2}. \end{aligned}$$

Exponentially damped sinusoids

1. $x(t) = e^{-bt}\cos(\omega_0 t)u(t)$

$$X(s) = \frac{s+b}{(s+b)^2 + \omega_0^2}$$

2. $x(t) = e^{-bt}\sin(\omega_0 t)u(t)$

$$X(s) = \frac{\omega_0}{(s+b)^2 + \omega_0^2}$$

Properties: differentiation in time domain

If $X(s) = \mathcal{L}[x(t)]$, then

$$\mathcal{L}\left[\frac{d}{dt}x(t)\right] = sX(s) - x(0),$$

(when $x(t)$ is discontinuous at $t = 0$ or if it contains an impulse or the derivative of an impulse at $t = 0$, $x(0)$ should be replaced by $x(0^-)$).

Proof: assume that, as $t \to \infty$, $x(t)$ decays faster than an exponential, i.e., $|x(t)| < Ce^{-at}$ for t large enough and for some $C \geq 0$ and $a > 0$. Then we integrate by parts to get

$$\begin{aligned} \mathcal{L}\left[\frac{d}{dt}x(t)\right] &= \int_0^\infty \left(\frac{d}{dt}x(t)\right) e^{-st} dt \\ &= \left[x(t)e^{-st}\right]_0^\infty + s\int_0^\infty x(t)e^{-st} dt \\ &= -x(0) + sX(s), \end{aligned}$$

where we have used the exponential decay of $x(t)$.

Differentiation in time domain

Ex.: $x(t) = \delta(t)$. Since $\delta(t) = (d/dt)u(t)$, we have

$$\mathcal{L}[\delta(t)] = s \cdot \frac{1}{s} - u(0^-) = 1.$$

For the second derivative, we have

$$\mathcal{L}\left[\frac{d^2}{dt^2}x(t)\right] = s^2 X(s) - sx(0) - x'(0),$$

where $x(0)$ and $x'(0)$ are, respectively, the values of $x(t)$ and $x'(t)$ at $t = 0$.

If $X(s) = \mathcal{L}[x(t)]$, then

$$\mathcal{L}\left[\int_0^\infty x(\lambda)d\lambda\right] = \frac{1}{s}X(s).$$

Proof: define the function

$$v(t) = \begin{cases} \int_0^t x(\lambda)d\lambda, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Then $v'(t) = x(t)$ for $t \geq 0$; since we are dealing with the one-sided Laplace transform, we assume that $x(t) = 0$ for $t < 0$. Hence, by the differentiation in time domain property we get

$$X(s) = sV(s),$$

so $V(s) = (1/s)X(s)$. **Q.E.D.**

Properties: convolution in time domain

If $X(s) = \mathcal{L}[x(t)]$ and $V(s) = \mathcal{L}[v(t)]$, then

$$\mathcal{L}[x(t) \star v(t)] = X(s)V(s).$$

Ex.: let $x(t) = v(t) = u(t) - u(t-1)$. Then

$$X(s) = \frac{1 - e^{-s}}{s}$$

and

$$\mathcal{L}[x(t) \star x(t)] = X^2(s) = \left(\frac{1 - e^{-s}}{s}\right)^2 = \frac{1 - 2e^{-s} + e^{-2s}}{s^2}.$$

We can actually invert this Laplace transform using its properties:

$$x(t) = tu(t) - 2(t-1)u(t-1) + (t-2)u(t-2).$$

Properties: initial-value theorem

Given a signal $x(t)$ with the Laplace transform $X(s)$, we can compute the initial values $x(0)$ and $x'(0)$ via

$$x(0) = \lim_{s \to \infty} sX(s)$$

and

$$x'(0) = \lim_{s \to \infty} [s^2 X(s) - sx(0)]$$

This property is called the **Initial-Value Theorem** (IVT).

Ex.: suppose the signal $x(t)$ has the Laplace transform

$$X(s) = \frac{-s^2 + 1}{2s^3 + 3s^2 + 1}.$$

Then

$$x(0) = \lim_{s \to \infty} sX(s) = \lim_{s \to \infty} \frac{-s^3 + s}{2s^3 + 3s^2 + 1} = -\frac{1}{2}.$$

Properties: final-value theorem

Given a signal $x(t)$, such that the limit

$$x(\infty) \triangleq \lim_{t \to \infty} x(t)$$

(called the **final value of** $x(t)$) exists, we have

$$x(\infty) = \lim_{s \to 0} sX(s).$$

Note that the limit on the right-hand side may exist, but it will only equal $x(\infty)$ if the latter limit is well-defined. This property is called the **Final-Value Theorem** (FVT).

Ex.: suppose

$$X(s) = \frac{1}{s^2 + 1}.$$

Then

$$\lim_{s \to 0} sX(s) = \lim_{s \to 0} \frac{s}{s^2 + 1} = 0.$$

However, $X(s) = \mathcal{L}[\sin(t)u(t)]$, and $\sin(t)u(t)$ does not have a final value.

The use of IVT and FVT

The IVT and the FVT are useful because they show how we can extract information about the initial and the final values of a signal from the knowledge of its Laplace transform. However, when using the FVT we should be careful to make sure that $x(\infty)$ is, in fact, defined.

Later we will see how to ascertain that from the form of the Laplace transform of $x(t)$.

We also note that the IVT cannot be used to infer $x(0^-)$, because the one-sided Laplace transform is based on the signal $x(t)$ for $t \geq 0$.