Plan for the lecture:

1. Recap: the Fourier transform
2. Motivation for the Laplace transform: exponential decaying factors
3. The Laplace transform: definition
4. Properties of the Laplace transform
Recap: The Fourier transform

We have seen earlier that we can represent sufficiently regular (e.g., absolutely integrable) signals via their Fourier transform:

\[ x(t) \leftrightarrow X(\omega) \]

\[ X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt \]

However, many important types of signals do not have a Fourier transform in the ordinary sense, i.e., the integral that defines \( X(\omega) \) diverges.

**Ex.:** \( x(t) = u(t) \)

\[ \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt = \int_{0}^{\infty} e^{-j\omega t} dt = \lim_{T \to \infty} \left[ -\frac{1}{j\omega} e^{-j\omega t} \right]_{0}^{T}, \]

and this limit does not exist because of the oscillatory behavior of \( e^{-j\omega t} \).
A possible way out

We can remedy this situation by introducing an exponentially decaying term $e^{-\sigma t}$, where $\sigma$ is a real constant.

Again, consider $x(t) = u(t)$.

\[
\int_{-\infty}^{\infty} u(t)e^{-\sigma t}e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(\sigma+j\omega)t} dt
\]
\[
= \lim_{T \to \infty} \left[ - \frac{1}{\sigma + j\omega} e^{-(\sigma+j\omega)t} \right]_{0}^{T}
\]
\[
= \frac{1}{\sigma + j\omega} - \frac{1}{\sigma + j\omega} \lim_{T \to \infty} \left[ e^{-\sigma T}e^{-j\omega T} \right].
\]

If $\sigma > 0$, then the above limit is equal to 0; if $\sigma \leq 0$, the limit does not exist. Thus, if $\sigma > 0$, we have

\[
\int_{-\infty}^{\infty} u(t)e^{-(\sigma+j\omega)t}dt = \frac{1}{\sigma + j\omega}.
\]
This leads us to the definition of the **Laplace transform**. Given a signal $x(t)$ and a complex number $s = \sigma + j\omega$, we define

$$X(s) \triangleq \int_{-\infty}^{\infty} x(t)e^{-st} \, dt = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t} \, dt.$$ 

The above integral may diverge for some values of $s$. The set of all complex numbers $s$ for which the above integral converges is called the **region of convergence (ROC)** of the Laplace transform of $x(t)$.

This is called the **bilateral** (or **two-sided**) Laplace transform. We can also define the **unilateral** (or **one-sided**) Laplace transform:

$$X(s) = \int_{0}^{\infty} x(t)e^{-st} \, dt.$$ 

We will deal with the one-sided Laplace transform, because that will allow us to deal conveniently with systems that have nonzero initial conditions.
The Laplace transform: an example

As an example, let us consider \( x(t) = e^{-bt}u(t) \), where \( b \) is real or complex.

\[
X(s) = \int_0^\infty x(t)e^{-st}dt
= \int_0^\infty e^{-(b+s)t}dt
= \lim_{T \to \infty} \left[ -\frac{1}{b+s} e^{-(b+s)t} \right]_0^T
= \frac{1}{b+s} - \frac{1}{b+s} \lim_{T \to \infty} \left[ e^{-(b+s)T} \right].
\]

The above limit will be equal to 0 if \( \text{Re}(b+s) > 0 \); otherwise, it does not exist. Hence,

\[
X(s) = \frac{1}{s+b}, \quad \text{if } \text{Re}(b+s) > 0.
\]

Thus, the ROC is the set of all complex numbers \( s \), such that \( \text{Re } s > -\text{Re } b \).
A word of caution: it is tempting to assume that we can compute the Fourier transform of $x(t)$ from its Laplace transform $X(s)$ by setting $s = j\omega$ (i.e., $\sigma = 0$). However, this is valid if and only if the set \{ $s \in \mathbb{C} : \text{Re} \ s = 0$ \} is in the ROC of $X(s)$.

Ex.: $x(t) = e^{-bt}u(t)$, $b$ real.

The ROC is \{ $s \in \mathbb{C} : \text{Re} \ s > -b$ \}. If $b > 0$, then the ROC includes all purely imaginary numbers. Thus,

$$X(\omega) = \left. \frac{1}{s + b} \right|_{s=j\omega} = \frac{1}{j\omega + b},$$

which we have proved before.

On the other hand, if $b \leq 0$, the ROC does not contain any purely imaginary number, so the Fourier transform does not exist, while the Laplace transform is equal to

$$X(s) = \frac{1}{s + b}$$

for any $s$ in the ROC.
Properties of the Laplace transform

Just as the Fourier transform, the Laplace transform has a number of useful properties which we will learn to exploit. We will look at and prove a few of these.

We will use the notation \( x(t) \leftrightarrow X(s) \) to denote the fact that \( X(s) \) is the Laplace transform of \( x(t) \); we will also use the notation \( X(s) = \mathcal{L}[x(t)] \).

First of all, the Laplace transform is linear: if \( X_1(s) = \mathcal{L}[x_1(t)] \) and \( X_2(s) = \mathcal{L}[x_2(t)] \), then for any constants \( a_1 \) and \( a_2 \)

\[
\mathcal{L}[a_1 x_1(t) + a_2 x_2(t)] = a_1 \mathcal{L}[x_1(t)] + a_2 \mathcal{L}[x_2(t)] = a_1 X_1(s) + a_2 X_2(s).
\]

This is obvious from definitions.
IF \( X(s) = \mathcal{L}[x(t)] \), then for any real number \( c > 0 \) we have

\[
x(t - c)u(t - c) \leftrightarrow e^{-cs} X(s).
\]

Proof:

\[
\mathcal{L}[x(t - c)u(t - c)] = \int_0^\infty x(t - c)u(t - c)e^{-st}dt
\]

\[
= \int_c^\infty x(t - c)e^{-st}dt
\]

\[
= \int_0^\infty x(t)e^{-s(t+c)}dt
\]

\[
= e^{-cs} \int_0^\infty x(t)e^{-st}dt
\]

\[
= e^{-cs} X(s).
\]

Q.E.D.
**Ex.:** consider the rectangular pulse

\[
x(t) = \begin{cases} 
  1, & 0 \leq t < c \\
  0, & \text{otherwise}
\end{cases}
\]

We can write

\[
x(t) = u(t) - u(t - c).
\]

Then, using linearity and shift property we get

\[
X(s) = \frac{1}{s} - \frac{e^{-cs}}{s} = \frac{1 - e^{-cs}}{s}.
\]

The ROC is \( \{s \in \mathbb{C} : \text{Re } s > 0\} \).
If $X(s) = \mathcal{L}[x(t)]$, then

\[
\mathcal{L}[t^N x(t)] = (-1)^N \frac{d^N}{ds^N} X(s)
\]

**Proof:** similar to the corresponding property of the Fourier transform.

**Ex.:** $x(t) = t^N u(t)$.

\[
X(s) = (-1)^N \frac{d^N}{ds^N} \left( \frac{1}{s} \right) = \frac{N!}{s^{N+1}}.
\]

In particular, for the unit ramp $x(t) = tu(t)$ we have

\[
X(s) = \frac{1}{s^2}.
\]

**Ex.:** $x(t) = te^{-bt} u(t)$.

\[
X(s) = (-1) \frac{d}{ds} \left( \frac{1}{s+b} \right) = \frac{1}{(s+b)^2}.
\]
Properties: multiplication by an exponential

If \( X(s) = \mathcal{L}[x(t)] \), then

\[
\mathcal{L}[e^{at} x(t)] = X(s - a)
\]

for any \( a \), real or complex.

**Proof:**

\[
\mathcal{L}[e^{at} x(t)] = \int_0^\infty x(t) e^{at} e^{-st} \, dt \\
= \int_0^\infty x(t) e^{-(s-a)t} \, dt \\
= X(s - a).
\]

**Ex.:** \( x(t) = [u(t) - u(t - c)] e^{at} \), where \( c > 0 \) and \( a \) is a real number. Then

\[
X(s) = \frac{1 - e^{-c(s-a)}}{s - a}.
\]
Laplace transform of cosine and sine

1. \( x(t) = \cos(\omega_0 t) u(t) = \frac{1}{2} \left[ e^{j\omega_0 t} + e^{-j\omega_0 t} \right] u(t) \)

\[
X(s) = \frac{1}{2} \left[ \frac{1}{s - j\omega_0} + \frac{1}{s + j\omega_0} \right] \\
= \frac{1}{2} \cdot \frac{2s}{s^2 + \omega_0^2} \\
= \frac{s}{s^2 + \omega_0^2}.
\]

2. \( x(t) = \sin(\omega_0 t) u(t) = \frac{1}{2j} \left[ e^{j\omega_0 t} - e^{-j\omega_0 t} \right] \)

\[
X(s) = \frac{1}{2j} \left[ \frac{1}{s - j\omega_0} - \frac{1}{s + j\omega_0} \right] \\
= \frac{1}{2j} \cdot \frac{2j\omega_0}{s^2 + \omega^2 + 0} \\
= \frac{\omega_0}{s^2 + \omega_0^2}.
\]
Exponentially damped sinusoids

1. \( x(t) = e^{-bt} \cos(\omega_0 t)u(t) \)

   \[ X(s) = \frac{s + b}{(s + b)^2 + \omega_0^2} \]

2. \( x(t) = e^{-bt} \sin(\omega_0 t)u(t) \)

   \[ X(s) = \frac{\omega_0}{(s + b)^2 + \omega_0^2} \]
If $X(s) = \mathcal{L}[x(t)]$, then

$$\mathcal{L}\left[ \frac{d}{dt} x(t) \right] = sX(s) - x(0),$$

(when $x(t)$ is discontinuous at $t = 0$ or if it contains an impulse or the derivative of an impulse at $t = 0$, $x(0)$ should be replaced by $x(0^-)$).

**Proof:** assume that, as $t \to \infty$, $x(t)$ decays faster than an exponential, i.e., $|x(t)| < Ce^{-at}$ for $t$ large enough and for some $C \geq 0$ and $a > 0$. Then we integrate by parts to get

$$\mathcal{L}\left[ \frac{d}{dt} x(t) \right] = \int_{0}^{\infty} \left( \frac{d}{dt} x(t) \right) e^{-st} dt$$

$$= [x(t)e^{-st}]_{0}^{\infty} + s \int_{0}^{\infty} x(t)e^{-st} dt$$

$$= -x(0) + sX(s),$$

where we have used the exponential decay of $x(t)$. 
Differentiation in time domain

**Ex.** \( x(t) = \delta(t) \). Since \( \delta(t) = (d/dt)u(t) \), we have

\[
\mathcal{L}[\delta(t)] = s \cdot \frac{1}{s} - u(0^-) = 1.
\]

For the second derivative, we have

\[
\mathcal{L} \left[ \frac{d^2}{dt^2} x(t) \right] = s^2 X(s) - sx(0) - x'(0),
\]

where \( x(0) \) and \( x'(0) \) are, respectively, the values of \( x(t) \) and \( x'(t) \) at \( t = 0 \).
If \( X(s) = \mathcal{L}[x(t)] \), then

\[
\mathcal{L} \left[ \int_0^\infty x(\lambda) d\lambda \right] = \frac{1}{s} X(s).
\]

**Proof:** define the function

\[
v(t) = \begin{cases} 
\int_0^t x(\lambda) d\lambda, & t \geq 0 \\
0, & t < 0
\end{cases}
\]

Then \( v'(t) = x(t) \) for \( t \geq 0 \); since we are dealing with the one-sided Laplace transform, we assume that \( x(t) = 0 \) for \( t < 0 \). Hence, by the differentiation in time domain property we get

\[
X(s) = sV(s),
\]

so \( V(s) = (1/s)X(s) \). Q.E.D.
If $X(s) = \mathcal{L}[x(t)]$ and $V(s) = \mathcal{L}[v(t)]$, then

\[ \mathcal{L}[x(t) \ast v(t)] = X(s)V(s). \]

**Ex.:** let $x(t) = v(t) = u(t) - u(t - 1)$. Then

\[ X(s) = \frac{1 - e^{-s}}{s} \]

and

\[ \mathcal{L}[x(t) \ast x(t)] = X^2(s) = \left(\frac{1 - e^{-s}}{s}\right)^2 = \frac{1 - 2e^{-s} + e^{-2s}}{s^2}. \]

We can actually invert this Laplace transform using its properties:

\[ x(t) = tu(t) - 2(t - 1)u(t - 1) + (t - 2)u(t - 2). \]
Properties: initial-value theorem

Given a signal \( x(t) \) with the Laplace transform \( X(s) \), we can compute the initial values \( x(0) \) and \( x'(0) \) via

\[
x(0) = \lim_{s \to \infty} sX(s)
\]

and

\[
x'(0) = \lim_{s \to \infty} [s^2X(s) - sx(0)]
\]

This property is called the **Initial-Value Theorem** (IVT).

**Ex.**: suppose the signal \( x(t) \) has the Laplace transform

\[
X(s) = \frac{-s^2 + 1}{2s^3 + 3s^2 + 1}.
\]

Then

\[
x(0) = \lim_{s \to \infty} sX(s) = \lim_{s \to \infty} \frac{-s^3 + s}{2s^3 + 3s^2 + 1} = -\frac{1}{2}.
\]
Properties: final-value theorem

Given a signal \( x(t) \), such that the limit

\[
x(\infty) \triangleq \lim_{t \to \infty} x(t)
\]

(called the **final value of** \( x(t) \)) exists, we have

\[
x(\infty) = \lim_{s \to 0} sX(s).
\]

Note that the limit on the right-hand side may exist, but it will only equal \( x(\infty) \) if the latter limit is well-defined. This property is called the **Final-Value Theorem** (FVT).

**Ex.**: suppose

\[
X(s) = \frac{1}{s^2 + 1}.
\]

Then

\[
\lim_{s \to 0} sX(s) = \lim_{s \to 0} \frac{s}{s^2 + 1} = 0.
\]

However, \( X(s) = \mathcal{L}[\sin(t)u(t)] \), and \( \sin(t)u(t) \) does not have a final value.
The IVT and the FVT are useful because they show how we can extract information about the initial and the final values of a signal from the knowledge of its Laplace transform. However, when using the FVT we should be careful to make sure that $x(\infty)$ is, in fact, defined.

Later we will see how to ascertain that from the form of the Laplace transform of $x(t)$.

We also note that the IVT cannot be used to infer $x(0^-)$, because the one-sided Laplace transform is based on the signal $x(t)$ for $t \geq 0$. 