Lecture XIII: Sampling

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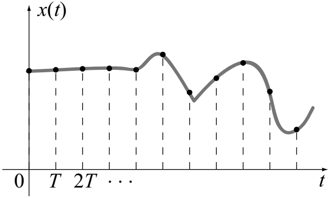
This lecture

Plan for the lecture:

1. Analog-to-digital conversion: sampling and quantization
2. Ideal sampling
   - frequency-domain representation
   - recovering bandlimited signals from samples
3. Shannon–Nyquist–Whittaker–Kotelnikov sampling theorem
4. Signal reconstruction from samples in time domain
   - the interpolation formula
   - the interpolating filter
5. Nonbandlimited signals and aliasing

Analog-to-digital conversion

Most signal processing these days is done digitally. For this reason, continuous-time signals must first be converted into digital signals, i.e., discrete-time signals that take on only a finite number of values. The first step in this analog-to-digital conversion is to *sample* the signal:



Given the sampling period $T > 0$, we convert a continuous-time signal $x(t)$ into a discrete-time signal $x[n]$, where

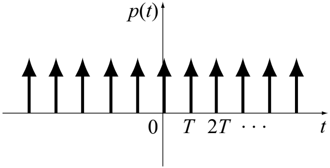
$$x[n] = x(nT), \qquad n = \dots, -2, -1, 0, 1, 2, \dots$$

Sampling is followed by *quantization*, i.e., the mapping of signal values $x[n]$ to a finite set of *quantization levels*. We will not consider quantization here.

Ideal sampling

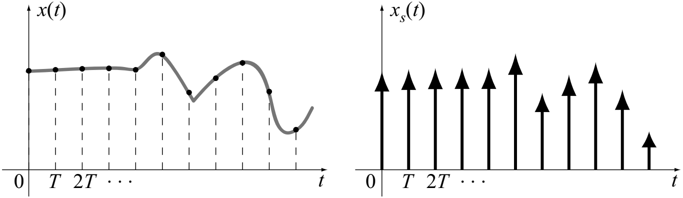
We will consider *ideal uniform sampling*. That is, the samples are T seconds apart, and there is no noise in the sampling process. To represent this kind of sampling mathematically, we can consider the impulse train

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT).$$



We can represent the sampled signal in continuous time by

$$x_s(t) = x(t)p(t)$$



Ideal sampling

Our goal is to prove that a bandlimited signal $x(t)$ can be reconstructed *exactly* from its sampled version $x_s(t)$, provided that sampling period T is sufficiently small.

As a first step, we will relate the Fourier transform $X_s(\omega)$ of $x_s(t)$ to that of $x(t)$. The pulse train $p(t)$ is T-periodic, so we can write it as a complex exponential Fourier series

$$p(t) = \sum_{=-\infty}^{\infty} c_k e^{jk\omega_s t},$$

where $\omega_s = 2\pi/T$ is the *sampling frequency* and

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} p(t) e^{-jk\omega_s t} dt = \sum_{n=-\infty}^{\infty} \frac{1}{T}\int_{-T/2}^{T/2} \delta(t - nT) e^{-jk\omega_s t} dt = \frac{1}{T}.$$

Thus,

$$p(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{jk\omega_s t}.$$

Ideal sampling in frequency domain

We have

$$x_s(t) = x(t)p(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} x(t)e^{jk\omega_s t}.$$

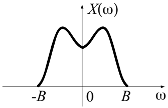
Therefore,

$$\begin{aligned} X_s(\omega) &= \mathcal{F}[x(t)p(t)] \\ &= \sum_{k=-\infty}^{\infty} \frac{1}{T}\mathcal{F}\left[x(t)e^{jk\omega_s t}\right] \\ &= \sum_{k=-\infty}^{\infty} \frac{1}{T}X(\omega - k\omega_s). \end{aligned}$$

Thus, the Fourier transform of the sampled signal $x_s(t)$ is a sum of scaled and shifted replicas of the Fourier transform of the original signal $x(t)$.

Bandlimited signals

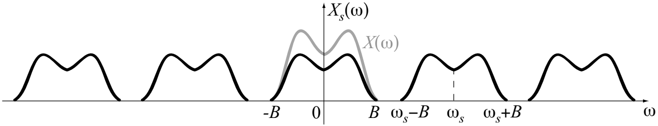
Assume that the input signal $x(t)$ is *bandlimited*, i.e., $|X(\omega)| = 0$ for $|\omega| > B$ with some $B > 0$. (We refer to B as the **signal bandwidth**.) For the sake of illustration, let's suppose that $X(\omega)$ looks like this:



Now let's suppose that the sampling frequency ω_s satisfies

$$\omega_s - B > B, \qquad \text{or } \omega_s > 2B$$

Then the scaled replicas of $X(\omega)$ shifted by the multiples of ω_s will not overlap:

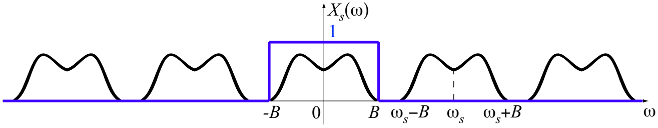


Recovering bandlimited signals

When the condition $\omega_s > 2B$ is satisfied, we can recover the original signal $x(t)$ from $x_s(t)$ by lowpass filtering and scaling:

$$X(\omega) = TH_{\rm LP}(\omega)X_s(\omega),$$

where the passband of the lowpass filter $H_{\rm LP}(\omega)$ is $|\omega| \le B$. We assume that $H_{\rm LP}(\omega) = p_{2B}(\omega)$, so that there is no phase distortion.



To see that this is true, we can write

$$\begin{aligned} TH_{\rm LP}(\omega)X_s(\omega) &= T \cdot \frac{1}{T} \sum_{k=-\infty}^{\infty} p_{2B}(\omega)X(\omega - k\omega_s) \\ &= X(\omega), \end{aligned}$$

since $p_{2B}(\omega)X(\omega - k\omega_s) = 0$ unless $k = 0$.

Shannon-Nyquist-Whittaker-Kotelnikov sampling theorem

We have the following important result, discovered in various forms by Shannon, Nyquist, Whittaker and Kotelnikov:

The Sampling Theorem

Consider a bandlimited signal $x(t)$ with bandwidth B. Then, provided the sampling frequency $\omega_s > 2B$, the signal $x(t)$ can be recovered exactly from its sampled version $x_s(t)$ by scaling and lowpass filtering:

$$X(\omega) = Tp_{2B}(\omega)X_s(\omega)$$

The minimum required sampling frequency $\omega_N \triangleq 2B$ is called the **Nyquist frequency** or the **Nyquist sampling rate**.

For example, recorded speech has essentially zero spectrum for all frequencies above $10\,\mathrm{kHz}$. Thus, the bandwidth is $2\pi \times 10^4\,\mathrm{rad/s}$, and the Nyquist rate is $4\pi \times 10^4\,\mathrm{rad/s}$. (For telephony applications, the Nyquist rate can be cut down to $8\,\mathrm{kHz}$.)

Signal reconstruction from samples: time domain

In the frequency domain, signal reconstruction from samples has the form

$$X(\omega) = Tp_{2B}(\omega)X_s(\omega).$$

In the time domain, this operation corresponds to convolving the sampled signal $x_s(t)$ with

$$h(t) = T\mathcal{F}^{-1}\left[p_{2B}(\omega)\right] = \frac{BT}{\pi}\mathrm{sinc}\left(\frac{Bt}{\pi}\right)$$

We can therefore write

$$\begin{aligned} x(t) &= x_s(t) \star h(t) \\ &= \left(\sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)\right) \star \frac{BT}{\pi}\mathrm{sinc}\left(\frac{Bt}{\pi}\right) \\ &= \frac{BT}{\pi}\sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT) \star \mathrm{sinc}\left(\frac{Bt}{\pi}\right) \\ &= \frac{BT}{\pi}\sum_{n=-\infty}^{\infty} x(nT)\mathrm{sinc}\left[\frac{B(t-nT)}{\pi}\right] \end{aligned}$$

The interpolation formula

We have derived another important result:

The Interpolation Formula

Let $x(t)$ be a bandlimited signal with bandwidth B. Provided the sampling frequency $\omega_s > 2B$, we can recover $x(t)$ from its samples $x(nT)$ via

$$x(t) = \frac{BT}{\pi} \sum_{n=-\infty}^{\infty} x(nT) \mathrm{sinc}\left[\frac{B(t-nT)}{\pi}\right],$$

where $T = 2\pi/\omega_s$ is the sampling period.

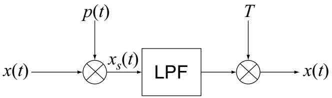
In other words, we can use the scaled and shifted sinc functions to recover the signal $x(t)$ by interpolating its sampled values $x(nT)$. For this reason, the filter with the impulse response

$$h(t) = \frac{BT}{\pi} \mathrm{sinc}\left(\frac{Bt}{\pi}\right)$$

is called the *interpolating filter*.

Sampling and reconstruction

The overall operation of sampling with subsequent reconstruction of a bandlimited signal $x(t)$ looks like this:

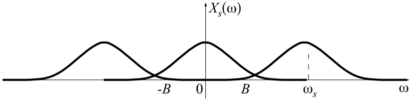


The sampled signal $x_s(t)$ can be stored in computer memory or in some other digital medium and then later recovered using the interpolating filter.

We are, of course, ignoring such effects as quantization errors (which are unavoidable in analog-to-digital conversion) or sampling jitter (when the locations of the samples fluctuate randomly around their nominal values). Another important consideration is that actual signals are never strictly bandlimited. For instance, an audio recording has a finite duration and therefore cannot be bandlimited. This can be handled via prefiltering.

Nonbandlimited signals and aliasing

Timelimited signals, strictly speaking, have infinite bandwidth. Thus, the scaled replicas of $X(\omega)$ that make up $X_s(\omega)$ will overlap no matter how large the sampling frequency ω_s is. This results in an effect called *aliasing*:



For audio signals, aliasing results in warbling at high frequencies. In image processing, aliasing leads to the so-called *moire effect*, i.e., periodic ringing.

When most of the signal spectrum (say, 90%) is concentrated in an interval $-B \leq \omega \leq B$ for some B, we can combat aliasing by prefiltering the signal with a lowpass filter to remove all frequencies $|\omega| > B$ and then sample. This will result in inexact signal recovery, but will get rid of aliasing.