Lecture XII: Ideal filters

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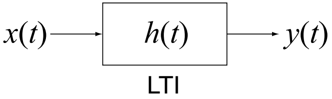
This lecture

Plan for the lecture:

1. LTI systems with sinusoidal inputs
2. Analog filtering
   - frequency-domain description: passband, stopband
   - amplitude and phase response of ideal filters
   - time-domain description
3. Detailed example: ideal lowpass filter
4. Nonideal filters

LTI systems with sinusoidal inputs

Consider an LTI system with impulse response $h(t)$:



$$y(t) = x(t) \star h(t) = \int_{-\infty}^{\infty} x(t-\lambda)h(\lambda)d\lambda$$

In the frequency domain we have

$$Y(\omega) = H(\omega)X(\omega),$$

where $X(\omega) = \mathcal{F}[x(t)], H(\omega) = \mathcal{F}[h(t)], Y(\omega) = \mathcal{F}[y(t)]$.

Consider a sinusoidal input of the form

$$x(t) = A\cos(\omega_0 t + \theta_0),$$

where A is the *amplitude*, ω_0 is the *frequency*, and θ_0 is the *phase*.

LTI systems with sinusoidal inputs

$$
\begin{aligned}
x(t) &= A\cos(\omega_0 t + \theta_0) \\
X(\omega) &= \pi A\left[e^{-j\theta_0}\delta(\omega+\omega_0) + e^{j\theta_0}\delta(\omega-\omega_0)\right]
\end{aligned}
$$

$$
\begin{aligned}
Y(\omega) &= H(\omega)X(\omega) \\
&= \pi A\left[e^{-j\theta_0}H(\omega)\delta(\omega+\omega_0) + e^{j\theta_0}H(\omega)\delta(\omega-\omega_0)\right] \\
&= \pi A\left[e^{-j\theta_0}H(-\omega_0)\delta(\omega+\omega_0) + e^{j\theta_0}H(\omega_0)\delta(\omega-\omega_0)\right] \\
&= \pi A\Big[e^{-j(\theta_0+\angle H(\omega_0))}|H(\omega_0)|\delta(\omega+\omega_0) \\
&\qquad + e^{j(\theta_0+\angle H(\omega_0))}|H(\omega_0)|\delta(\omega-\omega_0)\Big]
\end{aligned}
$$

$$
\begin{aligned}
y(t) &= \mathcal{F}^{-1}[Y(\omega)] \\
&= A|H(\omega_0)|\cos(\omega_0 t + \theta_0 + \angle H(\omega_0))
\end{aligned}
$$

Thus, the action of the LTI system with impulse response $h(t)$ on a sinusoid with amplitude A, frequency ω_0 and phase θ_0 is to transform the amplitude as $A \to A|H(\omega_0)|$ and the phase as $\theta_0 \to \theta_0 + \angle H(\omega_0)$.

LTI systems with sinusoidal inputs

Many signals encountered in practice are finite sums of sinusoids:

$$x(t) = \sum_{k=1}^{N} A_k \cos(\omega_k t + \theta_k)$$

The action of an LTI system with impulse response $h(t)$ on such an input is, by linearity, given by

$$y(t) = \sum_{k=1}^{N} A_k |H(\omega_k)| \cos(\omega_k t + \theta_k + \angle H(\omega_k))$$

Thus, an LTI system changes the amplitude ratios $\frac{A_k}{A_l}$ and the relative phases $\theta_k - \theta_l$ among the different frequency components $k, l = 1, \ldots, N$:

$$\begin{aligned} \frac{A_k}{A_l} &\quad\to\quad \frac{A_k}{A_l} \cdot \frac{H(\omega_k)}{H(\omega_l)} \\ \theta_k - \theta_l &\quad\to\quad \theta_k - \theta_l + \angle H(\omega_k) - \angle H(\omega_l) \end{aligned}$$

Analog filtering

These considerations naturally lead us to the notion of **filtering**: processing of signals in order to enhance certain frequency components and to reject certain others. For example, if a signal consists of a low-frequency information-bearing portion and a high-frequency noise portion, we can employ a filter to reject the high frequencies and thus remove the noise.

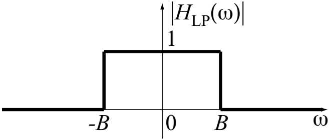
We will look at four kinds of filters:

1. **low-pass filters** pass all frequencies in the range $|\omega| \le B$, for some $B > 0$ and reject all others
2. **high-pass filters** pass all frequencies in the range $|\omega| \ge B$, for some $B > 0$ and reject all others
3. **bandpass filters** pass all frequencies in the range $B_1 \le |\omega| \le B_2$ for some $B_1, B_2 > 0$ with $B_1 < B_2$ and reject all others
4. **bandstop filters** pass all frequencies in the range $|\omega| \le B_1$ and $|\omega| \ge B_2$ for some $B_1, B_2 > 0$ with $B_1 < B_2$ and reject all others

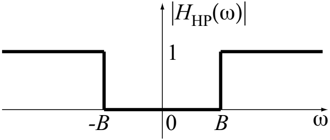
Analog filtering: frequency domain description

It is convenient to look at filters in the frequency domain. For each of the four kinds of filters, we will specify the amplitude response $|H(\omega)|$ and the phase response $\angle H(\omega)$.

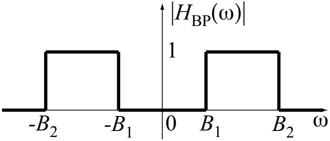
We start with the amplitude response. For the four filters we have defined above we have:



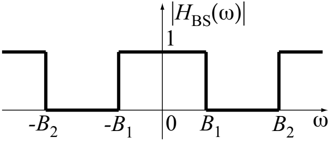
lowpass



highpass



bandpass



bandstop

Some filtering terminology

Given a filter $H(\omega)$, the set of frequencies ω such that $|H(\omega)| > 0$ is called the **passband** of the filter; the set of frequencies ω such that $|H(\omega)| = 0$ is called the **stopband** of the filter.

filter	passband	stopband
lowpass	$\lvert\omega\rvert \le B$	$\lvert\omega\rvert > B$
highpass	$\lvert\omega\rvert \ge B$	$\lvert\omega\rvert < B$
bandpass	$B_1 \le \lvert\omega\rvert \le B_2$	$\lvert\omega\rvert < B_1$ and $\lvert\omega\rvert > B_2$
bandstop	$\lvert\omega\rvert \le B_1$ and $\lvert\omega\rvert \ge B_2$	$B_1 < \lvert\omega\rvert < B_2$

If the input to a filter is a sinusoid $A\cos(\omega_0 t + \theta_0)$, then the amplitude of the output will be equal to:

- A, if the frequency ω_0 is in the passband of the filter
- 0, if the frequency ω_0 is in the stopband of the filter

Ideal filters

Next, we need to specify the phase response $\angle H(\omega)$ of the filter. We will call a filter $H(\omega)$ **ideal** if

$$|H(\omega)| = \begin{cases} 1, & \text{if } \omega \text{ is in the passband} \\ 0, & \text{if } \omega \text{ is in the stopband} \end{cases}$$

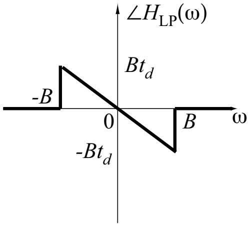
and

$$\angle H(\omega) = \begin{cases} -\omega t_d, & \text{if } \omega \text{ is in the passband} \\ 0, & \text{if } \omega \text{ is in the stopband} \end{cases}$$

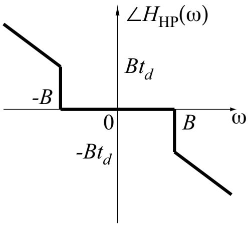
where $t_d > 0$ is some constant.

The reason for calling such filters "ideal" will become clear shortly.

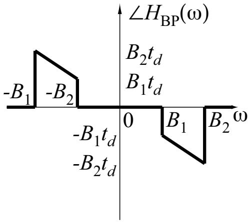
Phase response of ideal filters



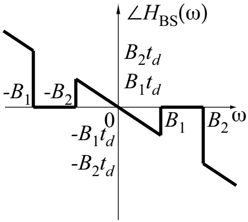
lowpass



highpass



bandpass



bandstop

Ideal filters with sinusoidal inputs

Let's see what happens when we feed a sinusoidal signal

$$x(t) = A\cos(\omega_0 t + \theta_0)$$

into an ideal filter $H(\omega)$. We have already seen that the output will be

$$y(t) = A|H(\omega_0)|\cos(\omega_0 t + \theta + \angle H(\omega_0)).$$

Since $|H(\omega)| = 1$ when ω is in the passband and 0 when ω is in the stopband, while $\angle H(\omega) = -\omega t_d$ when ω is in the passband and 0 otherwise, we can further write

$$y(t) = \begin{cases} A\cos(\omega_0(t - t_d) + \theta_0), & \text{if } \omega_0 \text{ is in the passband} \\ 0, & \text{if } \omega_0 \text{ is in the stopband} \end{cases}$$

In other words, if the frequency of the sinusoid ω_0 is in the passband of the filter, then the output $y(t)$ of the filter is a **time-delayed** version of the input $x(t)$:

$$y(t) = x(t - t_d).$$

Ideal filters with sinusoidal inputs

This explains why we use the term "ideal:" an ideal filter does not distort the input signal, only delays it (provided the input frequency is in the passband).

We can generalize these results to periodic signals that can be represented by sums of sinusoids,

$$x(t) = \sum_{k=1}^{\infty} A_k \cos(\omega_k t + \theta_t),$$

as well as to aperiodic signals that have a Fourier transform, $x(t) \leftrightarrow X(\omega)$. In the latter case, it is convenient to visualize the action of the filter in the frequency domain.

Detailed example: ideal lowpass filter

Let us consider in detail the lowpass filter whose amplitude and phase response are given by

$$|H_{\mathrm{LP}}(\omega)| = p_{2B}(\omega) \qquad \text{and} \qquad \angle H_{\mathrm{LP}}(\omega) = -\omega t_d p_{2B}(\omega),$$

where $p_{2B}(\omega)$ is a rectangle of unit height and width $2B$ centered at $\omega = 0$.

We have

$$H_{\mathrm{LP}}(\omega) = e^{-j\omega t_d} p_{2B}(\omega),$$

so that the impulse response has the form

$$h_{\mathrm{LP}}(t) = \mathcal{F}^{-1}\left[e^{-j\omega t_d} p_{2B}(\omega)\right] = \frac{B}{\pi}\mathrm{sinc}\left[\frac{B}{\pi}(t - t_d)\right]$$

Note that the frequency response $H_{\mathrm{LP}}(\omega)$ is bandlimited, hence the impulse response $h_{\mathrm{LP}}(t)$ cannot be timelimited. This implies that an ideal lowpass filter is **acausal** and therefore cannot be operated in real time.

Nonideal filters

In fact, it can be shown that any ideal filter is necessarily acausal, and therefore cannot be operated in real time. In practice, we have to resort to causal approximations of ideal filters. For example, an ideal lowpass filter can be approximated by an RC filter whose frequency response is described by

$$|H_{RC}(\omega)| = \frac{1}{\sqrt{(\omega RC)^2 + 1}} \qquad \text{and} \qquad \angle H_{RC}(\omega) = \tan^{-1}(-\omega RC)$$