Lecture X: Discrete-time Fourier transform

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This lecture

Plan for the lecture:

1. Recap: Fourier transform for continuous-time signals
2. Frequency content of discrete-time signals: the DTFT
3. Examples of DTFT
4. Inverse DTFT
5. Properties of the DTFT

Recap: Fourier transform

Recall from the last lecture that any sufficiently regular (e.g., finite-energy) continuous-time signal $x(t)$ can be represented in frequency domain via its Fourier transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt.$$

We can recover $x(t)$ from $X(\omega)$ via the inverse Fourier transform formula:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega.$$

Spectral content of discrete-time signals

In this lecture, we will look at one way of describing discrete-time signals through their frequency content: the discrete-time Fourier transform (DTFT).

Any discrete-time signal $x[n]$ that is **absolutely summable**, i.e.,

$$\sum_{n=-\infty}^{\infty} |x[n]| < +\infty,$$

has a DTFT $X(\Omega)$, $-\infty < \Omega < \infty$, given by

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\Omega}$$

Note that, even though the underlying signal $x[n]$ is discrete-time, the DTFT is a function of a **continuous** frequency Ω.

Periodicity of the DTFT

The first thing to note is that the DTFT $X(\Omega)$ of $x[n]$ is 2π-periodic:

$$\begin{aligned} X(\Omega + 2\pi) &= \sum_{n=-\infty}^{\infty} x[n] e^{-jn(\Omega+2\pi)} \\ &= \sum_{n=-\infty}^{\infty} x[n] e^{-jn\Omega} \underbrace{e^{-j2\pi n}}_{=1} \\ &= \sum_{n=-\infty}^{\infty} x[n] e^{-jn\Omega} \\ &= X(\Omega). \end{aligned}$$

This periodicity is due to the discrete-time nature of the signal. Thus, when working with DTFT's, we only need to look at the range $0 \le \Omega \le 2\pi$ (or $-\pi \le \Omega \le \pi$).

Computing DTFT's: an example

Consider

$$x[n] = \begin{cases} a^n, & q_1 \le n \le q_2 \\ 0, & \text{otherwise} \end{cases}$$

Then

$$\begin{aligned} X(\Omega) &= \sum_{n=q_1}^{q_2} a^n e^{-jn\Omega} \\ &= \sum_{n=q_1}^{q_2} (ae^{-j\Omega})^n \\ &= \frac{(ae^{-j\Omega})^{q_1} - (ae^{-j\Omega})^{q_2+1}}{1 - ae^{-j\Omega}} \end{aligned}$$

In the last step, we used the formula

$$\sum_{n=q_1}^{q_2} r^n = \frac{r^{q_1} - r^{q_2+1}}{1-r},$$

valid whenever q_1 and q_2 are integers with $q_2 > q_1$ and r is any real or complex number.

Computing DTFT's: another example

Consider the signal

$$x[n] = a^n u[n],$$

where $|a| < 1$. Then

$$\begin{aligned} X(\Omega) &= \sum_{n=0}^{\infty} a^n e^{-jn\Omega} \\ &= \sum_{n=0}^{\infty} (ae^{-j\Omega})^n \\ &= \frac{1}{1 - ae^{-j\Omega}}, \end{aligned}$$

where we used the formula

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r},$$

valid for any real or complex number r satisfying $|r| < 1$.

Computing DTFT's: another example

Consider the rectangular pulse

$$x[n] = \begin{cases} 1, & n = -q, -q+1, \ldots, q-1, q \\ 0, & \text{otherwise} \end{cases}$$

Then

$$\begin{aligned} X(\Omega) &= \sum_{n=-q}^{q} e^{-jn\Omega} \\ &= \frac{(e^{-j\Omega})^{-q} - (e^{-j\Omega})^{q+1}}{1 - e^{-j\Omega}} \\ &= \frac{e^{jq\Omega} - e^{-jq\Omega}e^{-j\Omega}}{1 - e^{-j\Omega}} \cdot \frac{e^{j\Omega/2}}{e^{j\Omega/2}} \\ &= \frac{e^{j(q+1/2)\Omega} - e^{-j(q+1/2)\Omega}}{e^{j\Omega/2} - e^{-j\Omega/2}} \\ &= \frac{\sin[(q+1/2)\Omega]}{\sin(\Omega/2)} \end{aligned}$$

Inverse DTFT

We can recover the original signal $x[n]$ from its DTFT $X(\Omega)$ via the inverse DTFT formula

$$x[n] = \frac{1}{2\pi} \int_0^{2\pi} X(\Omega) e^{jn\Omega} d\Omega.$$

Proof: use orthonormality of complex exponentials –

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} X(\Omega) e^{jn\Omega} d\Omega &= \frac{1}{2\pi} \int_0^{2\pi} \left(\sum_{m=-\infty}^{\infty} x[m] e^{-jm\Omega} \right) e^{jn\Omega} d\Omega \\
&= \sum_{m=-\infty}^{\infty} x[m] \cdot \underbrace{\frac{1}{2\pi} \int_0^{2\pi} e^{j(n-m)\Omega} d\Omega}_{=\delta[n-m]} \\
&= \sum_{m=-\infty}^{\infty} x[m] \delta[n-m], \\
&= x[n]
\end{aligned}$$

Properties of the DTFT

Like its continuous-time counterpart, the DTFT has several very useful properties. These are listed in any text on signals and systems. We will take a look at a couple of them.

First of all, the DTFT is linear: if

$$x_1[n] \leftrightarrow X_1(\Omega) \qquad \text{and} \qquad x_2[n] \leftrightarrow X_2(\Omega),$$

then

$$c_1 x_1[n] + c_2 x_2[n] \leftrightarrow c_1 X_1(\Omega) + c_2 X_2(\Omega)$$

for any two constants c_1, c_2.

The proof is obvious from definitions.

Convolution in time domain

If $x[n] \leftrightarrow X(\Omega)$ and $v[n] \leftrightarrow V(\Omega)$, then

$$x[n] \star v[n] \leftrightarrow X(\Omega)V(\Omega).$$

Proof: let $y[n] = x[n] \star v[n]$. Then

$$\begin{aligned}
Y(\Omega) &= \sum_{n=-\infty}^{\infty} (x[n] \star v[n]) e^{-jn\Omega} = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x[k]v[n-k] \right) e^{-jn\Omega} \\
&= \sum_{k=-\infty}^{\infty} x[k] \left(\sum_{n=-\infty}^{\infty} v[n-k] e^{-jn\Omega} \right) \\
&= \sum_{k=-\infty}^{\infty} x[k] \left(\sum_{n'=-\infty}^{\infty} v[n'] e^{-j(n'+k)\Omega} \right) \\
&= \underbrace{\left(\sum_{k=-\infty}^{\infty} x[k] e^{-jk\Omega} \right)}_{=X(\Omega)} \underbrace{\left(\sum_{n=-\infty}^{\infty} v[n] e^{-jn\Omega} \right)}_{=V(\Omega)}
\end{aligned}$$

Parseval's theorem

If $x[n]$ and $v[n]$ are real-valued signals, then

$$\sum_{n=-\infty}^{\infty} x[n]v[n] = \frac{1}{2\pi}\int_0^{2\pi} \overline{X(\Omega)}V(\Omega)d\Omega.$$

Proof:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} x[n]v[n] &= \sum_{n=-\infty}^{\infty} x[n]\left(\frac{1}{2\pi}\int_0^{2\pi} V(\Omega)e^{j\Omega n}d\Omega\right) \\
&= \frac{1}{2\pi}\int_0^{2\pi} V(\Omega)\left(\sum_{n=-\infty}^{\infty} x[n]e^{j\Omega n}\right)d\Omega \\
&= \frac{1}{2\pi}\int_0^{2\pi} V(\Omega)\underbrace{\left(\sum_{n=-\infty}^{\infty} x[n]e^{-j(-\Omega)n}\right)}_{=X(-\Omega)}d\Omega \\
&= \frac{1}{2\pi}\int_0^{2\pi} V(\Omega)\overline{X(\Omega)}d\Omega
\end{aligned}$$

where we used the fact that $x[n]$ is real-valued.

Parseval's theorem: cont'd

An important consequence of Parseval's theorem is that the signal energy

$$\sum_{n=-\infty}^{\infty} x^2[n]$$

can be computed also in the frequency domain:

$$\sum_{n=-\infty}^{\infty} x^2[n] = \frac{1}{2\pi} \int_0^{2\pi} |X(\Omega)|^2 d\Omega$$

Summary of the DTFT

The **discrete-time Fourier transform** (DTFT) gives us a way of representing frequency content of discrete-time signals.

The DTFT $X(\Omega)$ of a discrete-time signal $x[n]$ is a function of a **continuous** frequency Ω. One way to think about the DTFT is to view $x[n]$ as a sampled version of a continuous-time signal $x(t)$:

$$x[n] = x(nT), \qquad n = \ldots, -2, -1, 0, 1, 2, \ldots,$$

where T is a sufficiently small sampling step. Then $X(\Omega)$ can be thought of as a discretization of $X(\omega)$.

Due to discrete-time nature of the original signal, the DTFT is 2π-periodic. Hence, $\Omega = 2\pi$ is the highest frequency component a discrete-time signal can have.

The DTFT possesses several important properties, which can be exploited both in calculations and in conceptual reasoning about discrete-time signals and systems.