Lecture X: Discrete-time Fourier transform

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Plan for the lecture:

1. Recap: Fourier transform for continuous-time signals
2. Frequency content of discrete-time signals: the DTFT
3. Examples of DTFT
4. Inverse DTFT
5. Properties of the DTFT
Recall from the last lecture that any sufficiently regular (e.g., finite-energy) continuous-time signal $x(t)$ can be represented in frequency domain via its Fourier transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt.$$ 

We can recover $x(t)$ from $X(\omega)$ via the inverse Fourier transform formula:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega.$$
In this lecture, we will look at one way of describing discrete-time signals through their frequency content: the discrete-time Fourier transform (DTFT).

Any discrete-time signal $x[n]$ that is **absolutely summable**, i.e.,

$$\sum_{n=-\infty}^{\infty} |x[n]| < +\infty,$$

has a DTFT $X(\Omega)$, $-\infty < \Omega < \infty$, given by

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-jn\Omega}$$

Note that, even though the underlying signal $x[n]$ is discrete-time, the DTFT is a function of a **continuous** frequency $\Omega$. 
The first thing to note is that the DTFT $X(\Omega)$ of $x[n]$ is $2\pi$-periodic:

$$X(\Omega + 2\pi) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn(\Omega+2\pi)}$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-jn\Omega} e^{-j2\pi n} = 1$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-jn\Omega}$$

$$= X(\Omega).$$

This periodicity is due to the discrete-time nature of the signal. Thus, when working with DTFT’s, we only need to look at the range $0 \leq \Omega \leq 2\pi$ (or $-\pi \leq \Omega \leq \pi$).
Computing DTFT’s: an example

Consider

\[
x[n] = \begin{cases} 
a^n, & q_1 \leq n \leq q_2 \\
0, & \text{otherwise}
\end{cases}
\]

Then

\[
X(\Omega) = \sum_{n=q_1}^{q_2} a^n e^{-jn\Omega}
\]

\[
= \sum_{n=q_1}^{q_2} (ae^{-j\Omega})^n
\]

\[
= \frac{(ae^{-j\Omega})^{q_1} - (ae^{-j\Omega})^{q_2+1}}{1 - ae^{-j\Omega}}
\]

In the last step, we used the formula

\[
\sum_{n=q_1}^{q_2} r^n = \frac{r^{q_1} - r^{q_2+1}}{1 - r},
\]

valid whenever \(q_1\) and \(q_2\) are integers with \(q_2 > q_1\) and \(r\) is any real or complex number.
Consider the signal 
\[ x[n] = a^n u[n], \]
where \(|a| < 1\). Then
\[
X(\Omega) = \sum_{n=0}^{\infty} a^n e^{-jn\Omega} \\
= \sum_{n=0}^{\infty} (ae^{-j\Omega})^n \\
= \frac{1}{1 - ae^{-j\Omega}},
\]
where we used the formula
\[
\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r},
\]
valid for any real or complex number \(r\) satisfying \(|r| < 1\).
Computing DTFT’s: another example

Consider the rectangular pulse

\[ x[n] = \begin{cases} 
1, & n = -q, -q + 1, \ldots, q - 1, q \\
0, & \text{otherwise}
\end{cases} \]

Then

\[
X(\Omega) = \sum_{n=-q}^{q} e^{-jn\Omega} \\
= \frac{(e^{-j\Omega})^{-q} - (e^{-j\Omega})^{q+1}}{1 - e^{-j\Omega}} \\
= \frac{e^{jq\Omega} - e^{-jq\Omega} e^{-j\Omega}}{1 - e^{-j\Omega}} \cdot \frac{e^{j\Omega/2}}{e^{j\Omega/2}} \\
= \frac{e^{j(q+1/2)\Omega} - e^{-j(q+1/2)\Omega}}{e^{j\Omega/2} - e^{-j\Omega/2}} \\
= \frac{\sin[(q + 1/2)\Omega]}{\sin(\Omega/2)}
\]
Inverse DTFT

We can recover the original signal $x[n]$ from its DTFT $X(\Omega)$ via the inverse DTFT formula

$$x[n] = \frac{1}{2\pi} \int_{0}^{2\pi} X(\Omega) e^{jn\Omega} d\Omega.$$

**Proof:** use orthonormality of complex exponentials –

$$\frac{1}{2\pi} \int_{0}^{2\pi} X(\Omega) e^{jn\Omega} d\Omega = \frac{1}{2\pi} \int_{0}^{2\pi} \left( \sum_{m=-\infty}^{\infty} x[m] e^{-jm\Omega} \right) e^{jn\Omega} d\Omega$$

$$= \sum_{m=-\infty}^{\infty} x[m] \cdot \frac{1}{2\pi} \int_{0}^{2\pi} e^{j(n-m)\Omega} d\Omega$$

$$= \delta[n-m]$$

$$= \sum_{m=-\infty}^{\infty} x[m] \delta[n-m],$$

$$= x[n]$$
Properties of the DTFT

Like its continuous-time counterpart, the DTFT has several very useful properties. These are listed in any text on signals and systems. We will take a look at a couple of them.

First of all, the DTFT is linear: if

\[ x_1[n] \leftrightarrow X_1(\Omega) \quad \text{and} \quad x_2[n] \leftrightarrow X_2(\Omega), \]

then

\[ c_1 x_1[n] + c_2 x_2[n] \leftrightarrow c_1 X_1(\Omega) + c_2 X_2(\Omega) \]

for any two constants \( c_1, c_2 \).

The proof is obvious from definitions.
Convolution in time domain

If \( x[n] \leftrightarrow X(\Omega) \) and \( v[n] \leftrightarrow V(\Omega) \), then

\[
x[n] \ast v[n] \leftrightarrow X(\Omega)V(\Omega).
\]

**Proof:** let \( y[n] = x[n] \ast v[n] \). Then

\[
Y(\Omega) = \sum_{n=-\infty}^{\infty} (x[n] \ast v[n])e^{-jn\Omega} = \sum_{n=-\infty}^{\infty} \left( \sum_{k=-\infty}^{\infty} x[k]v[n-k] \right) e^{-jn\Omega}
\]

\[
= \sum_{k=-\infty}^{\infty} x[k] \left( \sum_{n=-\infty}^{\infty} v[n-k]e^{-jn\Omega} \right)
\]

\[
= \sum_{k=-\infty}^{\infty} x[k] \left( \sum_{n'=-\infty}^{\infty} v[n']e^{-j(n'+k)\Omega} \right)
\]

\[
= \left( \sum_{k=-\infty}^{\infty} x[k]e^{-jk\Omega} \right) \left( \sum_{n=-\infty}^{\infty} v[n]e^{-jn\Omega} \right)
\]

\[
= X(\Omega)V(\Omega)
\]
Parseval’s theorem

If $x[n]$ and $v[n]$ are real-valued signals, then

$$\sum_{n=-\infty}^{\infty} x[n]v[n] = \frac{1}{2\pi} \int_{0}^{2\pi} X(\Omega)V(\Omega) d\Omega.$$ 

**Proof:**

$$\sum_{n=-\infty}^{\infty} x[n]v[n] = \sum_{n=-\infty}^{\infty} x[n] \left( \frac{1}{2\pi} \int_{0}^{2\pi} V(\Omega)e^{j\Omega n} d\Omega \right)$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} V(\Omega) \left( \sum_{n=-\infty}^{\infty} x[n]e^{j\Omega n} \right) d\Omega$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} V(\Omega) \left( \sum_{n=-\infty}^{\infty} x[n]e^{-j(-\Omega)n} \right) d\Omega$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} V(\Omega) \overline{X(-\Omega)} d\Omega$$

where we used the fact that $x[n]$ is real-valued.
An important consequence of Parseval’s theorem is that the signal energy

$$\sum_{n=-\infty}^{\infty} x^2[n]$$

can be computed also in the frequency domain:

$$\sum_{n=-\infty}^{\infty} x^2[n] = \frac{1}{2\pi} \int_{0}^{2\pi} |X(\Omega)|^2 d\Omega$$
The **discrete-time Fourier transform** (DTFT) gives us a way of representing frequency content of discrete-time signals.

The DTFT $X(\Omega)$ of a discrete-time signal $x[n]$ is a function of a **continuous** frequency $\Omega$. One way to think about the DTFT is to view $x[n]$ as a sampled version of a continuous-time signal $x(t)$:

$$x[n] = x(nT), \quad n = \ldots, -2, -1, 0, 1, 2, \ldots,$$

where $T$ is a sufficiently small sampling step. Then $X(\Omega)$ can be thought of as a discretization of $X(\omega)$.

Due to discrete-time nature of the original signal, the DTFT is $2\pi$-periodic. Hence, $\Omega = 2\pi$ is the highest frequency component a discrete-time signal can have.

The DTFT possesses several important properties, which can be exploited both in calculations and in conceptual reasoning about discrete-time signals and systems.