BME 171, Sec 2: Homework 4 **due Tue, Oct 7 by 5pm**

Note: in some of these problems, you are asked to use MATLAB to plot certain signals. Turn in all your MATLAB code together with printouts of your plots with axes clearly labeled, just as in Kamen and Heck. Make sure that it is clear from the printout which plot corresponds to which problem.

1. **Computing Fourier series using Monte Carlo methods.** Computing Fourier coefficients of signals can be done only in very simple cases. In general, one has to resort to symbolic integration packages (such as Mathematica or MATLAB symbolic math toolbox) or use numerical techniques. In this exercise, you will learn how to use a very simple technique to get quick and dirty estimates of Fourier coefficients.

   Consider a signal $x(t)$ with period T that can be expanded in a trigonometric Fourier series

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t),$$

   where $\omega_0 = 2\pi/T$ is the fundamental frequency, and the coefficients $a_0, \{a_k\}, \{b_k\}$ are given by

$$\begin{aligned} a_0 &= \frac{1}{T}\int_{-T/2}^{T/2} x(t)dt, \\ a_k &= \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(k\omega_0 t)dt, \qquad k = 1, 2, \ldots \\ b_k &= \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(k\omega_0 t)dt, \qquad k = 1, 2, \ldots. \end{aligned}$$

   To approximate these coefficients, we can use the fact that, for any sufficiently "nice" function $f(t)$, the integral

$$\int_{-T/2}^{T/2} f(t)dt$$

   can be approximated by the sum

$$\frac{T}{N}\sum_{n=1}^{N} f\left(-\frac{T}{2} + \frac{(2n-1)T}{2N}\right), \tag{1}$$

   where N is some large positive integer. In other words, to approximate the integral of $f(t)$ over the interval $[-T/2, T/2]$, we carve that interval into N equal chunks, approximate the area under the graph of f over the nth chunk by

$$\text{(length of chunk)} \cdot \text{(value of } f \text{ at midpoint of chunk)} = \frac{T}{N} \cdot f\left(-\frac{T}{2} + \frac{(2n-1)T}{2N}\right),$$

   and then add these areas. Approximation methods of this kind are known as Monte Carlo methods.[1] Using the recipe in Equation (1), we can now compute Fourier coefficients of a given signal $x(t)$.

Consider a signal $x(t)$ which is a periodic extension of the function

$$s(t) = \begin{cases} -1 + \sqrt{-t}e^{t}, & -1 \leq t \leq 0 \\ 1 - \sqrt{t}e^{-t}, & 0 < t \leq 1 \end{cases}$$

Observe that $T = 2$. Given a positive integer m, let $x_m(t)$ denote the approximation to $x(t)$ obtained by using the first m terms in its Fourier series:

$$x_m(t) = a_0 + \sum_{k=1}^{m} a_k \cos(k\omega_0 t) + \sum_{k=1}^{m} b_k \sin(k\omega_0 t).$$

(a) Plot $x(t)$ for $-2T \leq t \leq 2T$.

(b) Using the Monte Carlo method of Equation (1) with $N = 200$, approximate $x_5(t)$, $x_{15}(t)$, and $x_{30}(t)$ and plot them for $-2T \leq t \leq 2T$ on separate sets of axes. (Hint: use symmetry properties of $x(t)$ to avoid doing unnecessary computation.)

(c) The squared error due to approximating $x(t)$ by $x_m(t)$ is given by

$$e_m = \frac{1}{T} \int_{-T/2}^{T/2} \Big(x(t) - x_m(t)\Big)^2 dt.$$

Use the Monte Carlo method with $N = 200$ to approximate e_1 through e_{30} and plot these values on a stem plot. How many coefficients do you need to get a squared error of less than 2 per cent of the maximum absolute value of $x(t)$?

2. **Fast decay of Fourier coefficients.** In this exercise, you will see how fast the Fourier coefficients of "nice" signals decay with k (the multiple of the fundamental frequency).

(a) Let $x(t)$ be a periodic signal with period T which is differentiable on the interval $(0, T)$. Prove that the Fourier coefficients $\{a_k\}$ and $\{b_k\}$ of $x(t)$ are such that

$$|a_k| \leq \frac{C_1}{k} \qquad \text{and} \qquad |b_k| \leq \frac{C_1}{k},$$

where $C_1 > 0$ is some constant that depends only on $x(t)$. (Hint: use the definitions of the Fourier coefficients and integrate by parts.)

(b) Now suppose that $x(t)$ is twice differentiable on the interval $(0, T)$. Prove that the Fourier coefficients $\{a_k\}$ and $\{b_k\}$ of $x(t)$ are such that

$$|a_k| \leq \frac{C_2}{k^2} \qquad \text{and} \qquad |b_k| \leq \frac{C_2}{k^2},$$

[1] More generally, in a Monte Carlo setup we would randonly draw N points $t_1, t_2, \ldots, t_N$ in the interval $[-T/2, T/2]$ and then approximate

$$\int_{-T/2}^{T/2} f(t)dt \approx \frac{T}{N} \sum_{n=1}^{N} f(t_n).$$

where $C_2 > 0$ is some constant that depends only on $x(t)$. (Hint: integrate by parts twice.)

Note: it may be tempting to assume that if a signal $x(t)$ is r times differentiable, then the absolute values of its Fourier coefficients decay as k^{-r}. However, that is not the case in general. Depending on the behavior of $x(t)$ at the boundary points $x = 0$ and $x = T$, it may or may not be possible to have faster than k^{-2} decay of the Fourier coefficients.

3. **Textbook problems.** Work problems 3.3(a,b,d,e) and 3.14 from Kamen and Heck.