BME 171, Sec 2: Homework 2 Solutions **due Tue, Sep 16 by 5pm**

1. Consider a system governed by the second-order differential equation

$$a\frac{d^2y(t)}{dt} + b\frac{dy(t)}{dt} + cy(t) = x(t),$$

where a, b and c are nonnegative real numbers.

(a) Show that this system is LTI. (5 points)

A: In simplest terms, the differential operator is an LTI operator. The differential equation given is simply a linear combination of applications of differential operators with constant coefficients; any system composed of a linear combination of LTI operators is itself an LTI. Thus, this system is an LTI.

More rigorously, you can show that the system is linear by proving additivity and homogeneity, and similarly prove that it is time invariant.

(b) Consider a complex exponential input $x(t) = e^{j\omega t}$. Show that the resulting output is of the form

$$y(t) = H(\omega)e^{j\omega t}$$

for some complex number $H(\omega)$. (Hint: try an output of the form $Ke^{j\omega t}$ and solve for K.) (10 points)

A:

$$\begin{aligned} y(t) &= Ke^{j\omega t} \\ y'(t) &= j\omega Ke^{j\omega t} \\ y''(t) &= -\omega^2 Ke^{j\omega t} \\ -a\omega^2 Ke^{j\omega t} + bj\omega Ke^{j\omega t} + cKe^{j\omega t} &= e^{j\omega t} \\ K &= \frac{1}{-a\omega^2 + bj\omega + c} \\ y(t) &= \frac{e^{j\omega t}}{-a\omega^2 + bj\omega + c} \end{aligned}$$

Thus the output is of the form $y(t) = H(\omega)e^{j\omega t}$.

(c) Consider now the sinusoidal input $x(t) = A\cos(\omega t + \theta)$ and express the resulting output as a sum of sinusoids with real coefficients. (10 points)

A: Using Euler's identity:

$$x(t) = A\cos(\omega t + \theta) = \frac{A}{2}(e^{j\theta}e^{j\omega t} + e^{-j\theta}e^{-j\omega t})$$

From above, we have the output to $e^{j\omega t}$. By inspection, the output to $e^{-j\omega t}$ is simply:

$$\frac{e^{-j\omega t}}{-a\omega^2 - bj\omega + c}$$

Thus, the output to x(t) is simply:

$$y(t) = \frac{A}{2}(e^{j\theta}\frac{e^{j\omega t}}{-a\omega^2 + bj\omega + c} + e^{-j\theta}\frac{e^{-j\omega t}}{-a\omega^2 - bj\omega + c})$$

Cross multiply and factor:

$$y(t) = \frac{A}{2}\frac{(-a\omega^2 + c)(e^{j\theta}e^{j\omega t} + e^{-j\theta}e^{-j\omega t}) - bj\omega(e^{j\theta}e^{j\omega t} - e^{-j\theta}e^{-j\omega t})}{a^2\omega^4 - 2a\omega^2 c + b^2\omega^2 + c^2}$$

Recall the formulae for sin and cos from HW 1, Problem 2B. Note that:

$$j(e^{j\theta}e^{j\omega t} - e^{-j\theta}e^{-j\omega t}) = -\frac{(e^{j\theta}e^{j\omega t} - e^{-j\theta}e^{-j\omega t})}{j} = -2\sin(\omega t + \theta)$$

Applying this and the formula for cos:

$$y(t) = A\frac{(-a\omega^2 + c)\cos(\omega t + \theta) + b\omega\sin(\omega t + \theta)}{a^2\omega^4 - 2a\omega^2 c + b^2\omega^2 + c^2}$$

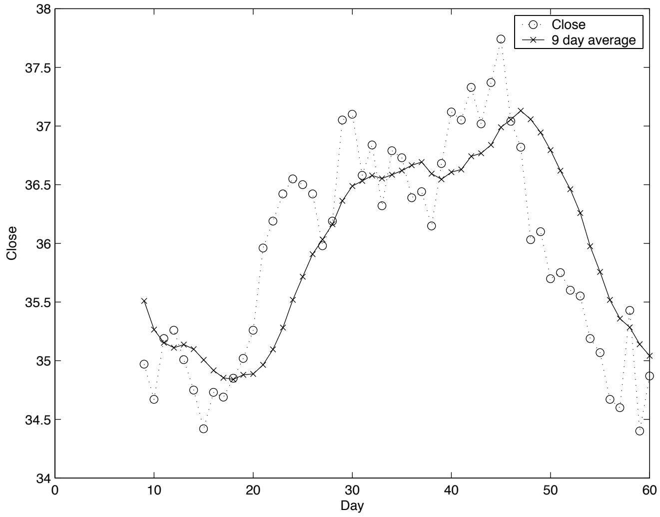
Thus, y(t) is a sum of sinuosoids with real coefficients.

This problem can also be solved by setting $y(t) = B\cos(\omega t + \theta) + C\sin(\omega t + \theta)$ and solving for B and C. You should come to the same solution.

2. **KH 1.16.** (5 points - 3 points for figure, 1 point for labels, 1 point for including code)

```
%% code to read data in not shown %%
n = 9:60
for k = n
avg(k-8) = 1/9 * sum(close(k-8:k));
end

plot(n,close(9:60),':o',n,avg,'-x')
xlabel('Day')
ylabel('Close')
legend('Close','9 day average')
```



3. **KH 1.17.** (12 points, 1 point per property per system. 2 additional bonus points available.)

   Systems a, b, c, and d are causal, as none of them depend on future values of x.

   Systems e and f are causal only if $t > 0$. If $t < 0$, these systems actually become acausal (2 bonus points for seeing this).

   Systems a, b, c and d are memoryless, as the current value of y does not depend on previous values of x.

   Systems e and f both have memory, as the integration of any function involving x(t) from 0 to t depends on previous values of x.

4. **KH 1.19.** (12 points, 2 per system)

   (a) is non-linear because $|x_1 + x_2| \neq |x_1| + |x_2|$ i.e. not additive

   (b) is non-linear because $e^{x_1+x_2} \neq e^{x_1} + e^{x_2}$ i.e. not additive

   (c) is linear because it is both homogenous and additive.

   (d) is non-linear because $ay(t) \neq ax(t)$ if $ax(t) > 10$. i.e. not homogeneous. This can also be shown to be non-additive.

   (e) and (f) are both linear because they only involve linear operations (integration and scalar multiplication).

5. **KH 1.20.** (12 points, 2 per system)

(a) and (b) are time-invariant because there is no dependence on the time of the input.

(c) is time varying, because:

$$\begin{aligned} S\{x(t-t_1)\} &= \sin(t)x(t-t_1) \\ y(t-t_1) &= \sin(t-t_1)x(t-t_1) \\ y(t-t_1) &\neq S\{x(t-t_1)\} \end{aligned}$$

(d) is time invariant because there is no dependence on the time of the input.

(e) is time varying because:

$$\begin{aligned} S\{x(\lambda-t_1)\} &= \int_0^t (t-\lambda)x(\lambda-t_1)d\lambda \\ y(t-t_1) &= \int_0^{t-t_1} (t-t_1-\lambda)x(\lambda)d\lambda \\ y(t-t_1) &\neq S\{x(\lambda-t_1)\} \end{aligned}$$

(f) is time varying because:

$$\begin{aligned} S\{x(t-t_1)\} &= \int_0^t \lambda x(\lambda-t_1)d\lambda \\ y(t-t_1) &= \int_0^{t-t_1} \lambda x(\lambda)d\lambda \\ y(t-t_1) &\neq S\{x(t-t_1)\} \end{aligned}$$

6. **KH 1.24.** (14 points - 3 points each for a, b and c. 5 points for part d)
(a)

$$\begin{aligned} x(t) &= 2u(t) - 2u(t-1) \\ y(t) &= 2(2(1-e^{-t}u(t)) - 2(2(1-e^{-(t-1)}u(t-1)) \\ y(t) &= 4(1-e^{-t})u(t) - 4(1-e^{-(t-1)})u(t-1) \end{aligned}$$

(b)

$$\begin{aligned} x(t) &= 4\cos(2(t-2)), \tau = 2(t-2) \\ x(\tau) &= 4\cos(\tau) \\ y(\tau) &= 4\sqrt{2}\cos(\tau - \frac{\pi}{4}) \\ y(t)) &= 4\sqrt{2}\cos(2t - 4 - \frac{\pi}{4}) \end{aligned}$$

(c)

$$\begin{aligned} x(t) &= 5u(t) + 10\cos(2t) \\ y(t) &= 5(2(1-e^{-t})u(t)) + 10(\sqrt{2}\cos(2t - \frac{\pi}{4}) \\ y(t) &= 10\left((1-e^{-t})u(t) + \sqrt{2}\cos(2t - \frac{\pi}{4})\right) \end{aligned}$$

(d) This one is tricky so pay attention, especially to how the integral of $2e^{-t}u(t)$ is computed.

$$\begin{aligned} x(t) &= tu(t) \\ x'(t) &= u(t) + t\delta(t) = u(t) \text{ (because } t\delta(t) = 0 \text{ for all t)} \\ y'(t) &= 2(1-e^{-t})u(t) = 2u(t) - 2e^{-t}u(t) \\ y(t) &= \int_0^t y'(\tau)d\tau = 2\int_0^t (1-e^{-\tau})u(\tau)d\tau \end{aligned}$$

(this is the case because $y(0) = 0$ as the system is linear)

Then you need to consider separately the cases when $t < 0$ and when $t \geq 0$. In the first case, the integrand is zero due to the unit step, so $y(t) = 0$ for $t < 0$. When $t \geq 0$, $u(\tau) = 1$ over the whole domain of integration, so you have:

$$y(t) = 2\int_0^t (1-e^{-\tau})d\tau = 2t + 2e^{-t} - 2.$$

Putting everything together, we get

$$y(t) = 2(t + e^{-t} - 1)u(t).$$

7. **KH 2.20** (20 points, 10 each)

(a) The voltage across the capacitor is y(t), the input voltage is x(t). Summing the voltages, we have:

$$\begin{aligned} x(t) &= y(t) + V_L(t) + V_R(t) \\ V_L(t) &= L\frac{di(t)}{dt} \\ i(t) &= C\frac{dy(t)}{dt} \\ V_L(t) &= L\frac{d}{dt}(C\frac{dy(t)}{dt}) = LC\frac{d^2y(t)}{dt^2} \\ V_R(t) &= Ri(t) = RC\frac{dy(t)}{dt} \\ x(t) &= LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) \end{aligned}$$

(b) The current is y(t), the input voltage is x(t). Summing the voltages, we have:

$$x(t) = V_C(t) + V_L(t) + V_R(t)$$

$$
\begin{aligned}
V_C(t) &= \frac{1}{C}\int_{-\infty}^{t} y(\tau)d\tau \\
V_L(t) &= L\frac{dy(t)}{dt} \\
V_R(t) &= Ri(t) = Ry(t) \\
x(t) &= \frac{1}{C}\int_{-\infty}^{t} y(\tau)d\tau + L\frac{dy(t)}{dt} + Ry(t) \\
\frac{dx(t)}{dt} &= L\frac{d^2y(t)}{dt^2} + R\frac{dy(t)}{dt} + \frac{y(t)}{C}
\end{aligned}
$$