

Quantum System Identification

Maxim Raginsky

Center for Photonic Communication and Computing
Department of Electrical and Computer Engineering

Northwestern University

2145 N. Sheridan Rd., Evanston, IL 60208-3118, USA

maxim@ece.northwestern.edu, <http://www.ece.northwestern.edu/~maxim>

Abstract

We formulate and study, in general terms, the problem of quantum system identification, i.e., the determination (or estimation) of unknown quantum channels through their action on suitably chosen input density operators. We also present a quantitative analysis of the worst-case performance of these schemes.

1 Introduction and background

In quantum information theory [1] all admissible devices are described mathematically by means of the so-called *quantum operations* (or *quantum channels*) [2, 3].

Given a complex Hilbert space \mathcal{H} , denote by $\mathcal{B}(\mathcal{H})$ the $*$ -algebra of all bounded operators on \mathcal{H} . In this paper we will work primarily with finite-dimensional Hilbert spaces, so that $\mathcal{B}(\mathcal{H})$ includes all linear operators on \mathcal{H} . Given Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 , a quantum channel T is a completely positive trace-preserving linear map of $\mathcal{B}(\mathcal{H}_1)$ into $\mathcal{B}(\mathcal{H}_2)$. All such maps admit the *Kraus decomposition* [3]

$$T(\rho) = \sum_k A_k \rho A_k^*, \quad (1)$$

where $A_k : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ are operators satisfying $\sum_k A_k^* A_k = I_{\mathcal{H}_1}$. This definition of the quantum channel is formulated in the Schrödinger picture, so that the density operators on \mathcal{H}_1 are mapped to density operators on \mathcal{H}_2 . The corresponding Heisenberg-picture definition goes the other way (observables on \mathcal{H}_2 are mapped to observables on \mathcal{H}_1) and yields a completely positive unit-preserving linear map $\hat{T} : \mathcal{B}(\mathcal{H}_2) \rightarrow \mathcal{B}(\mathcal{H}_1)$ related to T by the duality

$$\text{Tr}[T(\rho)X] = \text{Tr}[\rho\hat{T}(X)] \quad (2)$$

for all $\rho \in \mathcal{B}(\mathcal{H}_1)$ and all $X \in \mathcal{B}(\mathcal{H}_2)$. In this paper we will deal mostly with the Schrödinger picture.

This seemingly simple framework turns out to be rich enough to cover all kinds of general transformations of quantum-mechanical states. In fact, both purely classical and hybrid (classical-quantum or quantum-classical) transformations can be included as well, simply by restricting to a suitable Abelian subalgebra either at the input or at the output.

One of the basic challenges, both for theoreticians and for experimentalists, is to discover efficient procedures for analysis and synthesis of quantum channels. For instance, when designing a device for a specific task (e.g., an optimal quantum cloner [4]), one has to run tests in order to determine whether the device performs according to specification. Several such procedures have been proposed already, such as the tomographic scheme of D'Ariano and Lo Presti [5] or the maximum-likelihood reconstruction method of Ježek, Fiurášek, and Hradil [6].

All of these schemes rely, in one way or another, on the one-to-one correspondence [7] between completely positive maps $\mathcal{B}(\mathcal{H}_1) \rightarrow \mathcal{B}(\mathcal{H}_2)$ and positive operators on $\mathcal{H}_2 \otimes \mathcal{H}_1$, to which we shall return later in this paper. Our purpose here is to phrase the ideas common to these schemes as an abstract problem of *system identification*.

2 The quantum system identification problem

Consider the following arrangement, shown in Fig. 1: we are given a “black box” that implements an unknown quantum channel $T : \mathcal{B}(\mathcal{H}_1) \rightarrow \mathcal{B}(\mathcal{H}_2)$, which we need to determine. This will be done by presenting to the black box certain suitably chosen input density operators ρ , thereby obtaining output density operators $\sigma \equiv T(\rho)$, and then trying to determine (or to estimate) T given a set of ordered pairs $(\rho, T(\rho))$.

We assume that we can re-use the black box any fi-

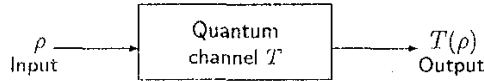


Figure 1: Input-output diagram for the quantum system identification problem.

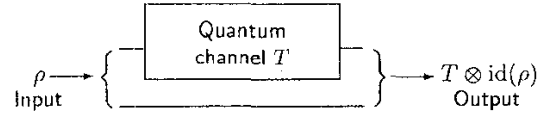


Figure 2: The set-up for the quantum system identification problem that allows for the use of quantum entanglement.

nite number of times, and that we can employ it as part of a more complicated arrangement. A typical strategy [5] is to use quantum entanglement [8]: one prepares input states (density operators) on the tensor product space $\mathcal{H}_1 \otimes \mathcal{H}_1$, and then subjects only one subsystem of the resulting composite system to T (see Fig. 2). This way we have the channel $T \otimes \text{id}$ from $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_1)$ into $\mathcal{B}(\mathcal{H}_2 \otimes \mathcal{H}_1)$, which corresponds uniquely to the original channel T . This extension is, in fact, at the basis of the Jamiolkowski isomorphism [7] (see next Section for a detailed discussion).

The problem of *quantum system identification* can now be formulated as follows. Instead of viewing the arrangement shown in Fig. 2 as a mapping of density operators ρ to density operators $T \otimes \text{id}(\rho)$, we can think of it as a mapping of quantum channels T into density operators $\rho[T]$ (we will use square brackets to distinguish the maps whose arguments are quantum channels from the maps that take density operators as arguments). That is, if we fix a density operator ρ , then we have $\rho[T] := T \otimes \text{id}(\rho)$. We will say that a density operator ρ is *admissible* if the map $\rho[\bullet]$ is invertible. Given an admissible density operator ρ , we will denote by ρ^\sharp the inverse mapping from density operators to channels.

This points, at least in principle, toward a solution of the problem of quantum system identification. All we need to do is to prepare an admissible state ρ , launch one of its subsystems through the black box T to get the output density operator $\sigma \equiv T \otimes \text{id}(\rho)$, and then reconstruct T as the inverse $\rho^\sharp(\sigma)$. Of course now we are faced with (at least) two more problems. (1) What states are admissible? (2) What can we say about the performance of the reconstruction procedure as a function of the (admissible) input state? We will address these problems in the remainder of this paper. The rest of this section will be devoted to discussion of the general properties of the map $\rho[\bullet]$.

We would like to make some statements about the continuity of $\rho[\bullet]$. Let us equip the algebra $\mathcal{B}(\mathcal{H})$ of linear operators on the Hilbert space \mathcal{H} with the *trace norm* [9], defined by $\|X\|_1 := \text{Tr}|X|$, where

the absolute value of an operator X is defined as $|X| := \sqrt{X^*X}$. Then $\|A\|_1 = \text{Tr} A$ for any positive operator A , and furthermore $\|\rho\|_1 = 1$ for any density operator ρ . We will also need to estimate norm differences of quantum channels; the ideal norm for this purpose is the so-called *norm of complete boundedness* (or CB-norm, for short) [10], defined by

$$\|T\|_{\text{cb}} := \sup_{n \in \mathbb{N}} \|T \otimes \text{id}_n\|, \quad \text{id}_n : \mathcal{M}_n \rightarrow \mathcal{M}_n \quad (3)$$

where \mathcal{M}_n stands for the algebra of $n \times n$ complex matrices. The norm $\|\bullet\|$ on the r.h.s. of (3) is the *operator norm*, defined for a general linear map $M : \mathcal{B}(\mathcal{H}_1) \rightarrow \mathcal{B}(\mathcal{H}_2)$ by

$$\|M\| := \sup_{X \in \mathcal{B}(\mathcal{H}_1); \|X\|_1 \leq 1} \|M(X)\|_1. \quad (4)$$

(Note that the above definition is tailored specifically for quantum channels in the Schrödinger picture; consult the monograph of Paulsen [10] for generalities.) We have $\|T\|_{\text{cb}} = 1$ for any quantum channel T . We shall have an occasion to use some other properties of the CB-norm in later sections; all we need right now is the inequality $\|T(A)\|_1 \leq \|T\|_{\text{cb}} \|A\|_1$, which is obvious from definitions, and the multiplicativity of the CB-norm with respect to the tensor product, $\|S \otimes T\|_{\text{cb}} = \|S\|_{\text{cb}} \|T\|_{\text{cb}}$.

With these preliminaries out of the way, consider a density operator ρ . We can easily see that the map $\rho[\bullet]$ is continuous. Indeed, consider two quantum channels, T and T' . By definition we have

$$\begin{aligned} \|\rho[T] - \rho[T']\|_1 &\equiv \|T \otimes \text{id}(\rho) - T' \otimes \text{id}(\rho)\|_1 \\ &\leq \|T \otimes \text{id} - T' \otimes \text{id}\|_{\text{cb}} \\ &= \|T - T'\|_{\text{cb}} \end{aligned} \quad (5)$$

(we have used the fact that the trace norm of a density operator is equal to one).

3 Admissible states and the Jamiolkowski isomorphism

In this section we describe an approach to the construction of admissible states. This method is closely connected to the Jamiolkowski isomorphism [7], and is, in fact, a natural extension of the latter. First we need some mathematical machinery from the theory of completely positive maps.

Given a C*-algebra \mathcal{A} , denote by \mathcal{A}^+ the cone of positive elements of \mathcal{A} (i.e., precisely those elements that can be written in the form A^*A for some $A \in \mathcal{A}$). A linear map T between C*-algebras \mathcal{A} and \mathcal{B} is called *positive* if $T(\mathcal{A}^+) \subseteq \mathcal{B}^+$, and *completely positive* (CP, for short) if the maps $T \otimes \text{id}_n : \mathcal{A} \otimes \mathcal{M}_n \rightarrow \mathcal{B} \otimes \mathcal{M}_n$ are positive for all $n \in \mathbb{N}$. A quantum channel is thus a specific instance of a CP map.

According to the fundamental theorem of Stinespring [11], for any CP map $T : \mathcal{B}(\mathcal{H}_1) \rightarrow \mathcal{B}(\mathcal{H}_2)$ there exist a Hilbert space \mathcal{E} and a bounded operator $V : \mathcal{H}_2 \rightarrow \mathcal{H}_1 \otimes \mathcal{E}$, such that

$$T(\rho) = V^*(\rho \otimes \mathbf{1}_{\mathcal{E}})V. \quad (6)$$

The pair (V, \mathcal{E}) is called the *Stinespring dilation* of T . Furthermore, with the additional requirement that the linear span of the set $\{(A \otimes \mathbf{1}_{\mathcal{E}})V\psi \mid A \in \mathcal{B}(\mathcal{H}_1), \psi \in \mathcal{H}_2\}$ be dense in $\mathcal{H}_1 \otimes \mathcal{E}$, the pair (V, \mathcal{E}) determines the CP map T uniquely up to unitary equivalence. In that case we speak of the *minimal Stinespring dilation* of T .

Consider now the set of all CP maps between $\mathcal{B}(\mathcal{H}_1)$ and $\mathcal{B}(\mathcal{H}_2)$, for some Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 . This set can be partially ordered in the following way. Given two CP maps T_1 and T_2 , we will say that T_1 is *completely dominated* by T_2 [12] (and write $T_1 \leq T_2$) if $T_2 - T_1$ is also a CP map. If $T_1 \leq \lambda T_2$ for some positive $\lambda \in \mathbb{R}$, we will say that T_1 is *completely λ -dominated* by T_2 .

Given a CP map T , the set of all CP maps completely dominated by T can be characterized completely using a theorem of Belavkin and Staszewski [12], which is referred to as the “Radon-Nikodym theorem” for CP maps, and asserts the following. Let (V, \mathcal{E}) be the minimal Stinespring dilation of a CP map T . Then a CP map S is completely λ -dominated by T if and only if it has the form

$$S(\rho) = V^*(\rho \otimes F)V \quad (7)$$

for some positive operator $F \in \mathcal{B}(\mathcal{E})$ with $\|F\| \leq \lambda$. Furthermore, the operator F is determined uniquely by S and (V, \mathcal{E}) .

Next we would like to show that the “Jamiolkowski isomorphism” between CP maps of $\mathcal{B}(\mathcal{H}_1)$ into

$\mathcal{B}(\mathcal{H}_2)$ and the positive operators on $\mathcal{H}_2 \otimes \mathcal{H}_1$ is a direct consequence of the above Radon-Nikodym theorem [13].

We will consider quantum channels from $\mathcal{B}(\mathcal{H}_1)$ into $\mathcal{B}(\mathcal{H}_2)$, where \mathcal{H}_1 and \mathcal{H}_2 are finite-dimensional Hilbert spaces. Let us fix an invertible density operator ρ on \mathcal{H}_1 , which we will call the *reference state*. Let p_i and ϕ_i be the eigenvalues and the eigenvectors of ρ , and let us also fix an orthonormal basis $\{f_\mu\}$ for \mathcal{H}_2 . Denoting by \mathcal{E} the tensor product $\mathcal{H}_2 \otimes \mathcal{H}_1$, define the isometry $V_\rho : \mathcal{H}_2 \rightarrow \mathcal{H}_1 \otimes \mathcal{E}$ by

$$V_\rho \psi := \sum_{i,\mu} p_i^{1/2} \langle f_\mu | \psi \rangle \phi_i \otimes f_\mu \otimes \phi_i. \quad (8)$$

Consider now a channel T . Let us define the unit vector $\Omega_\rho \in \mathcal{H}_1 \otimes \mathcal{H}_1$ by $\Omega_\rho := \sum_i p_i^{1/2} \phi_i \otimes \phi_i$ and the positive operator $F_{T,\rho}$ on \mathcal{E} by

$$F_{T,\rho} := (\mathbf{1} \otimes \rho^{-1})T \otimes \text{id}(|\Omega_\rho\rangle\langle\Omega_\rho|)(\mathbf{1} \otimes \rho^{-1}). \quad (9)$$

With these definitions, we can write

$$T(\sigma) = V_\rho^*(\sigma \otimes F_{T,\rho})V_\rho, \quad (10)$$

where the action of the coisometry V_ρ^* on the elementary tensors $\xi \otimes \eta \in \mathcal{H}_1 \otimes \mathcal{E}$ is given by

$$V_\rho^*(\xi \otimes \eta) = \sum_{i,\mu} p_i^{1/2} \langle f_\mu \otimes \phi_i | \eta \rangle \langle \phi_i | \xi \rangle f_\mu, \quad (11)$$

and then extended to all of $\mathcal{H}_1 \otimes \mathcal{E}$ by linearity.

It is now an easy consequence of the Belavkin-Staszewski theorem [12] that the operator $F_{T,\rho}$ uniquely determines the channel T , and that for any positive operator $F \in \mathcal{B}(\mathcal{E})$, the map

$$M(\sigma) = V_\rho^*(\sigma \otimes F)V_\rho \quad (12)$$

is completely positive. We see therefore that any invertible density operator ρ on \mathcal{H}_1 gives rise to an admissible pure state $\omega \equiv |\Omega_\rho\rangle\langle\Omega_\rho|$, in the sense that the mapping $\omega[T] := T \otimes \text{id}(|\Omega_\rho\rangle\langle\Omega_\rho|)$ is invertible. That is, the image of any density operator w on \mathcal{E} under the inverse map ρ^\sharp is given by

$$\rho^\sharp(w) = V_\rho^*(\bullet \otimes (\mathbf{1} \otimes \rho^{-1})w(\mathbf{1} \otimes \rho^{-1}))V_\rho. \quad (13)$$

It is important to realize that, in general, $\rho^\sharp(w)$ is *not* a quantum channel, unless w satisfies the additional consistency condition

$$\text{Tr}_{\mathcal{H}_2}[(\mathbf{1} \otimes \rho^{-1/2})w(\mathbf{1} \otimes \rho^{-1/2})] = \mathbf{1}_{\mathcal{H}_1}. \quad (14)$$

This will hold automatically in the quantum system identification setting (see Fig. 2), provided that there is no additional noise in the apparatus. We note also that the Jamiolkowski isomorphism is a special

case of this formalism [13], and is obtained if we pick as the reference state the maximally chaotic density operator $(\dim \mathcal{H}_1)^{-1} \mathbf{1}_{\mathcal{H}_1}$.

Before we go on, let us remark that the set of pure states on $\mathcal{H}_1 \otimes \mathcal{H}_1$ obtained by “purification” of invertible density operators on \mathcal{H}_1 does not exhaust all possibilities for admissible states. In a recent paper, D’Ariano and Lo Presti [14] have constructed a wide class of admissible states, which includes as a subset the states discussed here.

4 The performance of quantum system identification procedures

In this section we will quantify the performance of quantum system identification procedures as a function of the admissible state used as an input to the unknown channel.

Consider two density operators w_1 and w_2 on $\mathcal{H}_2 \otimes \mathcal{H}_1$ that satisfy the consistency condition (14). Then there exist quantum channels $T_1, T_2 : \mathcal{B}(\mathcal{H}_1) \rightarrow \mathcal{B}(\mathcal{H}_2)$ such that

$$w_i = \rho[T_i] \equiv T_i \otimes \text{id}(|\Omega_\rho\rangle\langle\Omega_\rho|), \quad i = 1, 2. \quad (15)$$

Furthermore, from (13) it follows that

$$T_i = \rho^\sharp(w_i) \equiv V_\rho^*(\bullet \otimes F_i)V_\rho, \quad (16)$$

where $F_i := (1 \otimes \rho^{-1})w_i(1 \otimes \rho^{-1})$, $i = 1, 2$. To determine how close the reconstructed channels $\rho^\sharp(w_1)$ and $\rho^\sharp(w_2)$ will be when the corresponding density operators w_1 and w_2 are close (say, in trace norm), we will get a lower bound on the *channel fidelity* [15] between $\rho^\sharp(w_1)$ and $\rho^\sharp(w_2)$, defined in the following way. Consider two channels $T_1, T_2 : \mathcal{B}(\mathcal{H}_1) \rightarrow \mathcal{B}(\mathcal{H}_2)$, and define the density operators

$$\sigma_i := T_i \otimes \text{id}(|\Omega\rangle\langle\Omega|), \quad (17)$$

where $\Omega := (\dim \mathcal{H}_1)^{-1/2} \sum_i e_i \otimes e_i$, the summation taken over some orthonormal basis of \mathcal{H}_1 . Note that $|\Omega\rangle\langle\Omega|$ is an admissible state corresponding to the maximally chaotic density operator $(\dim \mathcal{H}_1)^{-1} \mathbf{1}_{\mathcal{H}_1}$. Then the channel fidelity [15] is defined by

$$\mathcal{F}(T_1, T_2) := \left(\text{Tr} \sqrt{\sigma_1^{1/2} \sigma_2 \sigma_1^{1/2}} \right)^2, \quad (18)$$

where the quantity on the r.h.s. of (18) is the *mixed-state fidelity* [16]. We do not need all of the properties of the channel fidelity (18) [but see Ref. [15]], except the following:

$$2 - 2(\mathcal{F}(T_1, T_2))^{1/2} \leq \|\sigma_1 - \sigma_2\|_1, \quad (19)$$

which is a simple corollary of the results of Fuchs and van de Graaf [17]. We also note that the channel fidelity has the natural property that $\mathcal{F}(T_1, T_2) = 1$ if and only if $T_1 \equiv T_2$ (this is a straightforward consequence of the properties of the mixed-state fidelity [16]).

We can rewrite w_1 and w_2 from (15) in terms of ρ , σ_1 , and σ_2 :

$$\sigma_i = \frac{(1 \otimes \rho^{-1/2})w_i(1 \otimes \rho^{-1/2})}{\dim \mathcal{H}_1}, \quad i = 1, 2. \quad (20)$$

Then we can use the well-known inequalities $\|AB\|_1 \leq \|A\|\|B\|_1$ and $\|A^*A\| = \|A\|^2$, where $\|\bullet\|$ is the usual operator norm [9], to get

$$\|\sigma_1 - \sigma_2\|_1 \leq \frac{\|\rho^{-1}\| \cdot \|w_1 - w_2\|_1}{\dim \mathcal{H}_1}. \quad (21)$$

Combining this estimate with (19); we obtain

$$\begin{aligned} 2 - 2\sqrt{\mathcal{F}(\rho^\sharp(w_1), \rho^\sharp(w_2))} \\ \leq \frac{\|\rho^{-1}\| \cdot \|w_1 - w_2\|_1}{\dim \mathcal{H}_1}. \end{aligned} \quad (22)$$

Upon rearranging, we get the desired lower bound:

$$\begin{aligned} \mathcal{F}(\rho^\sharp(w_1), \rho^\sharp(w_2)) \\ \geq \left(1 - \frac{\|\rho^{-1}\|}{2 \dim \mathcal{H}_1} \|w_1 - w_2\|_1 \right)^2. \end{aligned} \quad (23)$$

We see right away that worst-case performance of the channel reconstruction procedure is controlled by the smallest eigenvalue of ρ (or, equivalently, by the largest eigenvalue of ρ^{-1}). This fact has also been pointed out by D’Ariano and Lo Presti [5], and Eq. (23) gives the corresponding quantitative estimate. Note that in the case of $\rho = (\dim \mathcal{H}_1)^{-1} \mathbf{1}_{\mathcal{H}_1}$, the constant in front of the trace norm on the r.h.s. of (23) is 1/2, which yields worst-case performance that depends only on the states w_1 and w_2 .

Note that we have discussed here the ideal scenario, namely that there is no additional noise in the apparatus used for the channel reconstruction. Any such disturbance will, of course, further degrade the performance of the scheme.

5 Discussion and conclusions

We have outlined a general mathematical framework for quantum system identification, i.e., the determination (or estimation) of quantum channels through their action on suitably chosen input density operators (we have called them *admissible states*). In

general, the channel reconstruction procedure will involve the preparation of an *entangled state*, followed by the application of an unknown channel to one of the subsystems, leaving the other one intact. One can show [5, 14, 18] that the use of entangled states results in an overall improvement, in either the precision or the stability of the reconstruction procedure. On a more fundamental level, however, the use of entanglement is also essential in light of the one-to-one correspondence between quantum channels and bipartite density operators [that satisfy the consistency condition (14)], which can be explained in abstract terms within the framework of Radon-Nikodym type theorems for CP maps [12, 13].

In this paper we have emphasized quantum channels acting on finite-dimensional algebras, in order to keep the presentation simple. However, it is important (e.g., for quantum information-theoretic studies in quantum optics) to have a mathematical theory of quantum system identification in infinite dimensions. Some steps in this direction have already been taken (see, e.g., D'Ariano and Lo Presti [14] or Raginsky [13]). Let us briefly comment on some of the big points. One starts with a density operator ρ that is invertible; however, the inverse is now an *unbounded* operator. This implies that the reconstruction map ρ^\sharp will fail to be continuous, which will result in an unbounded growth of statistical errors during the tomographic estimation of the matrix elements of the Radon-Nikodym density $F_{T,\rho}$.

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