

LETTER TO THE EDITOR

Entropy production rates of bistochastic strictly contractive quantum channels on a matrix algebra

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Abstract

We derive, for a bistochastic strictly contractive quantum channel on a matrix algebra, a relation between the contraction rate and the rate of entropy production. We also sketch some applications of our result to the statistical physics of irreversible processes and to quantum information processing.

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Let \mathcal{A} be the algebra of observables (say, a C^* -algebra with identity), associated with a quantum-mechanical system Σ . A general evolution of Σ is described, in the Heisenberg picture, by a map $T : \mathcal{A} \rightarrow \mathcal{A}$ which is (i) unital: $T(\mathbb{1}) = \mathbb{1}$, and (ii) completely positive: for any nonnegative integer n , the map $T \otimes \text{id} : \mathcal{A} \otimes \mathcal{M}_n \rightarrow \mathcal{A} \otimes \mathcal{M}_n$, where \mathcal{M}_n is the algebra of $n \times n$ complex matrices, sends positive operators to positive operators [1]. We shall henceforth refer to such maps as (quantum) channels [2]. If Σ is an N -level system, then its algebra of observables is isomorphic to \mathcal{M}_N , which is exclusively the case we shall consider in this letter. A celebrated result of Kraus [1] then says that, for any channel T , there exists a collection of at most N^2 operators $V_i \in \mathcal{M}_N$, which we shall call the Kraus operators associated with T , such that (i) $T(A) = \sum_i V_i^* A V_i$, and (ii) $\sum_i V_i^* V_i = \mathbb{1}$. It is now easy to see that $T(A^*) = T(A)^*$ for all $A \in \mathcal{M}_N$, i.e., any channel maps Hermitian operators to Hermitian operators.

Given a channel T , the corresponding Schrödinger-picture channel \hat{T} is defined via the duality

$$\text{tr}[\hat{T}(A)B] = \text{tr}[AT(B)]$$

whence it follows that \hat{T} is a completely positive map which preserves the trace, i.e., $\text{tr}\hat{T}(A) = \text{tr}A$ for all $A \in \mathcal{M}_N$. In other words, \hat{T} maps the set \mathcal{D}_N of $N \times N$ density matrices into itself. Furthermore, in terms of the Kraus operators V_i , we have $\hat{T}(A) = \sum_i V_i A V_i^*$, so that $\hat{T}(A^*) = \hat{T}(A)^*$ as well.

The set $\{T^n\}_{n \in \mathbb{N}}$ is a discrete-time quantum-dynamical semigroup generated by T , i.e., $T^n T^m = T^{n+m}$ and we take $T^0 \equiv \text{id}$ (the identity channel). It is easy to show that, for any channel T , there exists at least one density operator ρ such that $\hat{T}(\rho) = \rho$ [3]. A question of

clear physical importance is to determine whether the dynamics generated by T is relaxing [3], i.e., whether there exists a density operator ρ such that, for any density operator σ , the orbit $\{\hat{T}^n(\sigma)\}$ converges to ρ in the trace norm $\|A\|_1 := \text{tr}(A^*A)^{1/2}$.

One way to show that a dynamics is relaxing relies on the so-called Liapunov's direct method [4]. Let \mathcal{X} be a compact separable space, and let $\varphi : \mathcal{X} \rightarrow \mathcal{X}$ be a continuous map, such that

- (i) φ has a unique fixed point $x_0 \in \mathcal{X}$, and
- (ii) there exists a Liapunov function for φ , i.e., a continuous functional S on \mathcal{X} such that, for all $x \in \mathcal{X}$, $S[\varphi(x)] \geq S(x)$, where equality holds if and only if $x \equiv x_0$.

Then, for any $x \in \mathcal{X}$, the sequence $\{\varphi^n(x)\}$ converges to x_0 .

Let T be a bistochastic channel, i.e., one for which $T(\mathbb{1}) = \hat{T}(\mathbb{1}) = \mathbb{1}$. If we treat \mathcal{M}_N as a Hilbert space with the Hilbert–Schmidt inner product, $\langle A, B \rangle := \text{tr}(A^*B)$, then an easy calculation shows that the Schrödinger-picture channel \hat{T} is precisely the adjoint of T with respect to $\langle \cdot, \cdot \rangle$, i.e., $\langle A, T(B) \rangle = \langle \hat{T}(A), B \rangle$ for all $A, B \in \mathcal{M}_N$. The composite map $T \circ \hat{T}$ (which we shall write henceforth as $T\hat{T}$) is also a bistochastic channel, which is, furthermore, a Hermitian operator with respect to $\langle \cdot, \cdot \rangle$. In [5], Streater proved the following result.

Theorem 1. *Let $T : \mathcal{M}_N \rightarrow \mathcal{M}_N$ be a bistochastic channel. Suppose that \hat{T} is ergodic with a spectral gap $\gamma \in [0, 1)$, i.e., (i) up to a scalar multiple, the identity matrix $\mathbb{1}$ is the only fixed point of \hat{T} in all of \mathcal{M}_N , and (ii) the spectrum of $T\hat{T}$ is contained in the set $[0, 1 - \gamma] \cup \{1\}$. Then, for any $\sigma \in \mathcal{D}_N$, we have*

$$S[\hat{T}(\sigma)] - S(\sigma) \geq \frac{\gamma}{2} \|\sigma - N^{-1}\mathbb{1}\|_2^2 \quad (1)$$

where $S(\sigma) := -\text{tr}(\sigma \ln \sigma)$ is the von Neumann entropy of σ and $\|A\|_2 := [\text{tr}(A^*A)]^{1/2}$ is the Hilbert–Schmidt norm of A .

In other words, if a bistochastic channel \hat{T} is ergodic, then the dynamics generated by T is relaxing by Liapunov's theorem¹. Furthermore, the relaxation process is accompanied by entropy production at a rate controlled by the spectral gap.

Now we have an interesting 'inverse' problem. Consider a bistochastic channel T on \mathcal{M}_N with \hat{T} strictly contractive [6]. That is, \hat{T} is uniformly continuous on \mathcal{D}_N (in the trace-norm topology) with Lipschitz constant $C \in [0, 1)$: for any pair $\sigma, \sigma' \in \mathcal{D}_N$, we have $\|\hat{T}(\sigma) - \hat{T}(\sigma')\|_1 \leq C\|\sigma - \sigma'\|_1$. Then by the contraction mapping principle [7], $N^{-1}\mathbb{1}$ is the only density matrix left invariant by \hat{T} , and furthermore $\|\hat{T}(\sigma) - N^{-1}\mathbb{1}\|_1 \rightarrow 0$ as $n \rightarrow \infty$ for any $\sigma \in \mathcal{D}_N$, i.e., the dynamics generated by T is relaxing. The question is, does the entropy-gain estimate (1) hold, and, if so, how does the spectral gap γ depend on the contraction rate C ?

This problem was motivated in the first place by the following observation. In the case of \mathcal{M}_2 , the action of a bistochastic strictly contractive channel \hat{T} can be given a direct geometric interpretation. Recall that the density matrices in \mathcal{M}_2 are in a one-to-one correspondence with the points of the closed unit ball in \mathbb{R}^3 . Then the image of \mathcal{D}_2 under a strictly contractive channel \hat{T} with contraction rate C will be contained inside the closed ball of radius C , centred at the origin [6], i.e., the image of \mathcal{D}_2 under \hat{T} will consist only of mixed states. This geometric illustration suggests that the rate of entropy increase under \hat{T} must be related to the contraction rate. Now even though in the case of \mathcal{M}_N with $N \geq 3$ we no longer have such a convenient

¹ Endowing the set \mathcal{D}_N with the trace-norm topology takes care of all the continuity requirements imposed by Liapunov's theorem.

geometric illustration, nevertheless it seems plausible that the rate of entropy production under a bistochastic strictly contractive channel would still be controlled by the contraction rate.

Indeed it turns out that the contraction rate is related to the rate of entropy production, as stated in the following theorem.

Theorem 2. *Let T be a bistochastic channel on \mathcal{M}_N , such that \hat{T} is strictly contractive with rate C . Then \hat{T} is ergodic with spectral gap $\gamma \geq 1 - C$, so that, for any $\sigma \in \mathcal{D}_N$, we have*

$$S[\hat{T}(\sigma)] - S(\sigma) \geq \frac{1 - C}{2} \|\sigma - N^{-1} \mathbb{1}\|_2^2. \quad (2)$$

Proof. We first prove that \hat{T} is ergodic. As we noted before, T and \hat{T} are adjoints of each other with respect to the Hilbert–Schmidt inner product. Using the Kadison–Schwarz inequality [8]

$$\Phi(A^*A) \geq \Phi(A)^*\Phi(A)$$

for any channel Φ on a C^* -algebra \mathcal{A} , as well as the fact that

$$\mathrm{tr}T(A) = \mathrm{tr}[\hat{T}(\mathbb{1})A] = \mathrm{tr}A$$

for any $A \in \mathcal{M}_N$, we find that

$$\|T(A)\|_2^2 = \mathrm{tr}[T(A)^*T(A)] \leq \mathrm{tr}[T(A^*A)] = \mathrm{tr}(A^*A) = \|A\|_2^2$$

and the same goes for \hat{T} . That is, both T and \hat{T} are contractions on \mathcal{M}_N (in the Hilbert–Schmidt norm), hence their fixed-point sets coincide [9].

By hypothesis, \hat{T} leaves invariant the density matrix $N^{-1}\mathbb{1}$, which is invertible. In this case a theorem of Fannes, Nachtergaele, and Werner [10, 11] says that $T(X) = X$ if and only if $V_i X = X V_i$ for all V_i , where V_i are the Kraus operators associated with T . It was shown in [6] that if \hat{T} is strictly contractive, then the set of all X such that $V_i X = X V_i$ for all V_i consists precisely of multiples of the identity matrix. We see, therefore, that $T(X) = X$ if and only if $X = \chi \mathbb{1}$ for some $\chi \in \mathbb{C}$, whence it follows that $\hat{T}(X) = X$ if and only if X is a multiple of $\mathbb{1}$. This proves ergodicity of \hat{T} .

Our next task is to establish the spectral gap estimate $\gamma \geq 1 - C$. Let X be a Hermitian operator with $\mathrm{tr}X = 0$. In that case we can find a density operator ρ and a sufficiently small number $\epsilon > 0$ such that $\sigma := \rho + \epsilon X$ is still a density operator [3].² Then

$$\|\hat{T}(X)\|_1 = (1/\epsilon) \|\hat{T}(\sigma) - \hat{T}(\rho)\|_1 \leq (C/\epsilon) \|\sigma - \rho\|_1 = C\|X\|_1. \quad (3)$$

Because one is a simple eigenvalue of both T and \hat{T} , it is also a simple eigenvalue of $T\hat{T}$. Hence $1 - \gamma$ (which we may as well assume to belong to the spectrum of $T\hat{T}$) is the largest eigenvalue of the restriction of $T\hat{T}$ to traceless matrices. Let Y be the corresponding eigenvector. Without loss of generality we may choose Y to be Hermitian³. Then, using equation (3) and the fact that $\|\Phi(A)\|_1 \leq \|A\|_1$ for any trace-preserving completely positive map Φ [13], we may write

$$(1 - \gamma)\|Y\|_1 = \|T\hat{T}(Y)\|_1 \leq \|\hat{T}(Y)\|_1 \leq C\|Y\|_1$$

which yields the desired spectral gap estimate. The entropy gain bound (2) now follows from theorem 1. \square

² This may be seen as a simple consequence of the following fact [12]. The set $\mathcal{D}_N^{\mathrm{inv}}$ of all invertible $N \times N$ density matrices is a smooth manifold, where the tangent space at any $\rho \in \mathcal{D}_N^{\mathrm{inv}}$ can be naturally identified with the set of $N \times N$ traceless Hermitian matrices.

³ Recall that $1 - \gamma$ is real, and $T\hat{T}(A^*) = [T\hat{T}(A)]^*$ for all A , which implies that $Y + Y^*$ is also an eigenvector of $T\hat{T}$ with the same eigenvalue.

Remark. Equation (3) can also be proved using the following finite-dimensional specialization of a general result due to Ruskai [13]. If $T : \mathcal{M}_N \rightarrow \mathcal{M}_N$ is a channel, then

$$\sup_{A=A^*; \text{tr}A=0} \frac{\|\hat{T}(A)\|_1}{\|A\|_1} = \frac{1}{2} \sup_{\psi, \phi \in \mathbb{C}^N; \langle \psi | \phi \rangle = 0} \|\hat{T}(|\psi\rangle\langle\psi| - |\phi\rangle\langle\phi|)\|_1. \quad (4)$$

Because \hat{T} is strictly contractive, the right-hand side of (4) is bounded from above by C , and (3) follows. The supremum on the left-hand side of (4) is the ‘Dobrushin contraction coefficient’, studied extensively by Lesniewski and Ruskai [14] in connection with the contraction of monotone Riemannian metrics on quantum state spaces under (duals of) quantum channels.

Note that in some cases the sharper estimate

$$S[\hat{T}(\sigma)] - S(\sigma) \geq \frac{1 - C^2}{2} \|\sigma - N^{-1}\mathbb{1}\|_2^2 \quad (5)$$

may be shown to hold. Consider, for instance, the case $T = \hat{T}$, so that the eigenvalues of \hat{T} are all real. Let $\lambda_1, \dots, \lambda_L$, $L = N^2 - 1$, be the eigenvalues of \hat{T} that are distinct from unity. Then we claim that $\max_j |\lambda_j| \leq C$, which can be proved via *reductio ad absurdum*. Suppose that there exists some X (which we may take to be Hermitian) with $\text{tr}X = 0$ such that $\hat{T}(X) = \lambda X$ with $|\lambda| > C$. We may use the same trick as in the proof above to show that there exist two density operators, σ and ρ , such that $\|\hat{T}(\sigma) - \hat{T}(\rho)\|_1 > C\|\sigma - \rho\|_1$, which would contradict the strict contractivity of \hat{T} . Because $T\hat{T} = \hat{T}^2$, we have $1 - \gamma = (\max_j |\lambda_j|)^2 \leq C^2$, which confirms (5). Furthermore, using a theorem of King and Ruskai [15], the bound (5) can be established for *all* bistochastic strictly contractive channels on \mathcal{M}_2 , as well as for tensor products of such channels.

The proof of this last assertion goes as follows. Let T be a channel on \mathcal{M}_2 such that \hat{T} is strictly contractive. Then \hat{T} is ergodic, the proof of which can be taken *verbatim* from the proof of theorem 2. It is left to show that $1 - \gamma \leq C^2$. Any density operator in \mathcal{M}_2 can be written as

$$\rho = \frac{1}{2} \left(\mathbb{1} + \sum_{j=1}^3 r_j \sigma_j \right) \quad (6)$$

where the σ_j are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the real numbers r_j satisfy the condition $r_1^2 + r_2^2 + r_3^2 \leq 1$ (this is precisely the one-to-one correspondence between \mathcal{D}_2 and the closed unit ball in \mathbb{R}^3). The King–Ruskai theorem [15] asserts that, for any bistochastic channel T on \mathcal{M}_2 , there exist unitaries U, V and real numbers ξ_j , $1 \leq j \leq 3$, with $|\xi_j| \leq 1$ such that, for any $\rho \in \mathcal{D}_2$,

$$\hat{T}(\rho) = U[\hat{T}_{\text{diag}}(V\rho V^*)]U^*$$

where the action of the map \hat{T}_{diag} on the density operator (6) is given by

$$\hat{T}_{\text{diag}}(\rho) = \frac{1}{2} \left(\mathbb{1} + \sum_{j=1}^3 \xi_j r_j \sigma_j \right).$$

It can easily be shown [6] that if \hat{T} is strictly contractive, then $C = \max_j |\xi_j|$.

The parameters ξ_j are determined as follows [15]. Consider the orthonormal basis of \mathcal{M}_2 , generated by $\mathbb{1}$ and the Pauli matrices, with respect to which \hat{T} and T can be written as 4×4 matrices in the block-diagonal form

$$\hat{T} = \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix} \quad T = \begin{pmatrix} 1 & 0 \\ 0 & M^t \end{pmatrix}$$

where M^t denotes the transpose of M^4 . Then the absolute values of the parameters ξ_j are precisely the singular values of M . This implies that $C = \|M\|$, where $\|M\|$ denotes the operator norm (the largest singular value) of M . Consequently we get $1 - \gamma = \|M^*M\| = \|M\|^2 \equiv C^2$, which yields the entropy gain estimate (5). The argument for tensor products of bistochastic channels on \mathcal{M}_2 runs along similar lines.

We remark that the results reported in this letter are consistent with the following theorem [3]. Let T be a channel on \mathcal{M}_N with the property that \hat{T} has a unique fixed point $\rho \in \mathcal{D}_N$ in all of \mathcal{M}_N . Let λ_j be the eigenvalues of \hat{T} distinct from one, and let $\kappa := \max_j |\lambda_j|$. Then there exist a polynomial p and an N -dependent constant K such that, for any $\sigma \in \mathcal{D}_N$,

$$\|\hat{T}^n(\sigma) - \rho\|_1 \leq K p(n) \kappa^n. \quad (7)$$

This shows that the dynamics generated by T is relaxing, $\|\hat{T}^n(\sigma) - \rho\|_1 \rightarrow 0$ as $n \rightarrow \infty$, and that the rate of convergence is controlled essentially by the eigenvalue of \hat{T} with the second largest modulus. Now, if T is a bistochastic channel with \hat{T} strictly contractive, then it follows from theorem 2 that $\kappa \leq C^{1/2}$. But by virtue of strict contractivity we have

$$\|\hat{T}^n(\sigma) - N^{-1}\mathbb{1}\|_1 \equiv \|\hat{T}^n(\sigma) - \hat{T}^n(N^{-1}\mathbb{1})\|_1 \leq C^n \|\sigma - N^{-1}\mathbb{1}\|_1 < \frac{2C^{n/2}(N-1)}{N}$$

which has the form of (7). To obtain the last inequality we used the fact $C < C^{1/2}$ for $0 \leq C < 1$, as well as the fact that the set \mathcal{D}_N is compact and convex, so that the convex functional $\|\sigma - N^{-1}\mathbb{1}\|_1$ attains its supremum on an extreme point of \mathcal{D}_N , i.e., on a pure state. In turn, a routine calculation shows [6] that, for any pure state σ , $\|\sigma - N^{-1}\mathbb{1}\|_1 = 2(N-1)/N$.

Finally, a few comments are in order as to how our results come to bear upon (a) the statistical physics of irreversible processes [4] and (b) quantum information processing [16].

In the first setting we are interested in a concise mathematical description of the approach to equilibrium. We consider a quantum-mechanical system with the Hilbert space \mathcal{H} , $\dim \mathcal{H} \leq \infty$. We will use \mathcal{A} to denote the algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on \mathcal{H} . Let H be the (possibly unbounded) Hamiltonian of the system, with the additional requirement that the operator $e^{-\beta H}$ is trace-class for all positive real β . This means that the Gibbs state exists for all positive inverse temperatures, and that each eigenvalue of H has finite multiplicity. Let us write down the spectral decomposition $H = \sum_E P_E$, where E are the eigenvalues of H and P_E are the corresponding eigenprojections. Then $n_E := \dim P_E \mathcal{H} < \infty$ for all E . Suppose we are given a channel $T : \mathcal{A} \rightarrow \mathcal{A}$ such that, for each E , the restriction T_E of T to the algebra $\mathcal{A}_E := P_E \mathcal{A} P_E$ is bistochastic, i.e., $T(P_E) = \hat{T}(P_E) = P_E$, so that $T(\mathcal{A}_E) \subseteq \mathcal{A}_E$, and strictly contractive. Then theorem 2 says that, for each E , equation (2) holds with some contraction rate C_E . The Hilbert spaces $P_E \mathcal{H}$ can be thought of as the energy levels of H , and the states P_E/n_E as the corresponding microcanonical states. If in addition we can show that $C := \sup_E C_E < 1$, then any normal state of the entire system will converge towards the Gibbs state with the rate C .

In the framework of quantum information processing we are concerned mostly with systems that have finite-dimensional Hilbert spaces. We may envision either a quantum register that is used for storing information, or a quantum computer that processes information.

⁴ The 3×3 matrix M must be real because both \hat{T} and T map Hermitian operators to Hermitian operators.

In both of these situations we are interested in stabilizing information against the effects of decoherence. The decoherence mechanism is modelled by a channel. It was shown in [6] that the set of strictly contractive channels on an algebra \mathcal{A} is dense in the set of all channels on \mathcal{A} (here we are talking about the Schrödinger-picture channels). The same result also holds for bistochastic strictly contractive channels, which are dense in the set of all bistochastic channels [6]. For such channels theorem 2 may be used to estimate the rate at which entropy is produced in the register or in the computer. Such estimates generally serve as a measure of efficiency of error-correction procedures [17] (note that it was shown in [6] that errors modelled by strictly contractive channels can be corrected only approximately).

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