

Radon–Nikodym derivatives of quantum operations

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Given a completely positive (CP) map T , there is a theorem of the Radon–Nikodym type [W. B. Arveson, *Acta Math.* **123**, 141 (1969); V. P. Belavkin and P. Staszewski, *Rep. Math. Phys.* **24**, 49 (1986)] that completely characterizes all CP maps S such that $T-S$ is also a CP map. This theorem is reviewed, and several alternative formulations are given along the way. We then use the Radon–Nikodym formalism to study the structure of order intervals of quantum operations, as well as a certain one-to-one correspondence between CP maps and positive operators, already fruitfully exploited in many quantum information-theoretic treatments. We also comment on how the Radon–Nikodym theorem can be used to derive norm estimates for differences of CP maps in general, and of quantum operations in particular. © 2003 American Institute of Physics. [DOI: 10.1063/1.1615697]

I. INTRODUCTION

In the mathematical framework of quantum information theory,¹ all admissible devices are modeled by the so-called quantum operations^{2,3}—that is, completely positive linear contractions on the algebra of observables of the physical system under consideration. Thus it is of paramount importance to have at one’s disposal a good analysis toolkit for completely positive (CP) maps.

There are many useful structure theorems for CP maps. The two best known ones, due to Stinespring⁴ and Kraus,³ are *de rigueur* in virtually all quantum information-theoretic treatments. These theorems are significant because each of them states that a given map is CP if and only if it is expressible in a certain canonical form. However, in many applications we need to consider whole families of CP maps. This necessitates the introduction of comparison tools for CP maps, e.g., when the family of CP maps in question admits some sort of (partial) order.

Mathematically, the set of all CP maps between two algebras of observables is a cone that can be partially ordered in the following natural way. If S and T are two CP maps, we write $S \leq T$ if $T-S$ is CP as well. This partial order comes up in, e.g., the problem of distinguishing between two known CP maps with given *a priori* probabilities under the constraint that the average probability of error is minimized.⁵ A typical way of dealing with partially ordered cones is to exhibit a correspondence between the cone’s order and a partial order of some “simpler” objects. This is accomplished by means of theorems of the Radon–Nikodym type, as in the case of, e.g., partial ordering of positive measures or positive linear functionals. There are a number of Radon–Nikodym theorems for CP maps (see, e.g., the work of Arveson,⁶ Belavkin and Staszewski,⁷ Davies,² Holevo,⁸ Ozawa,⁹ and Parthasarathy¹⁰) that differ widely in scope and in generality. Thus, the results of Davies, Ozawa, and Holevo have to do with Radon–Nikodym derivatives of CP instruments⁹ with respect to scalar measures. On the other hand, ideas common to the Arveson and Belavkin–Staszewski theorems, with further developments by Parthasarathy, are directly applicable to the partial ordering of CP maps described above, and will therefore be the focus of the present article. More specifically, we will demonstrate that certain problems encountered in quantum information-theoretic settings that involve characterization and comparison of CP maps, are best understood in this Radon–Nikodym framework.

The paper is organized as follows. We summarize the salient facts on CP maps and quantum

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operations in Sec. II. In Sec. III we review the Arveson–Belavkin–Staszewski formulation of the Radon–Nikodym theorem for CP maps and state several alternative, but equivalent, versions. The Radon–Nikodym machinery is then applied to the following problems: partial ordering of quantum operations (Sec. IV), characterization of quantum operations by means of positive operators (Sec. V), and estimating norms of differences of CP maps (Sec. VI). Finally some concluding remarks are made in Sec. VII.

II. PRELIMINARIES

A. Completely positive maps

1. Definitions

Let \mathcal{A} and \mathcal{B} be C^* -algebras; denote by \mathcal{A}^+ the cone of positive elements of \mathcal{A} . A linear map $T: \mathcal{A} \rightarrow \mathcal{B}$ is called *positive* if $T(\mathcal{A}^+) \subseteq \mathcal{B}^+$. Given some $n \in \mathbb{N}$, let \mathcal{M}_n be the algebra of $n \times n$ complex matrices. The map T is called *n-positive* if the induced map $T \otimes \text{id}_n: \mathcal{A} \otimes \mathcal{M}_n \rightarrow \mathcal{B} \otimes \mathcal{M}_n$ is positive, and *completely positive* if it is *n-positive* for all $n \in \mathbb{N}$.

One typically considers maps $T: \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$, where \mathcal{A} is a C^* -algebra with identity, and $\mathcal{B}(\mathcal{H})$ is the algebra of bounded operators on a complex separable Hilbert space \mathcal{H} . Then it can be shown⁴ that T is CP if and only if, for each $n \in \mathbb{N}$,

$$\sum_{i,j=1}^n \langle \eta_i | T(A_i^* A_j) \eta_j \rangle \geq 0 \quad \forall \eta_i \in \mathcal{H}, \quad A_i \in \mathcal{A}; \quad i=1, \dots, n. \quad (1)$$

2. Theorems of Stinespring and Kraus

A fundamental theorem of Stinespring⁴ states that, for any normal (i.e., ultraweakly continuous) CP map $T: \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$, there exist a Hilbert space \mathcal{K} , a $*$ -homomorphism $\pi: \mathcal{A} \rightarrow \mathcal{B}(\mathcal{K})$, and a bounded operator $V: \mathcal{H} \rightarrow \mathcal{K}$, such that

$$T(A) = V^* \pi(A) V \quad \forall A \in \mathcal{A}. \quad (2)$$

We will refer to any such triple (\mathcal{K}, V, π) [or, through a slight abuse of language, to the form (2) of T] as a *Stinespring dilation* of T . Given T , one can construct its Stinespring dilation in such a way that $\mathcal{K} = \pi(\mathcal{A})V\mathcal{H}$, i.e., the set $\{\pi(A)V\psi | A \in \mathcal{A}, \psi \in \mathcal{H}\}$ is total in \mathcal{K} . With this additional property, the Stinespring dilation is unique up to unitary equivalence,¹¹ and is called the *minimal Stinespring dilation*.

For the special case of a CP map $T: \mathcal{B}(\mathcal{H}_1) \rightarrow \mathcal{B}(\mathcal{H}_2)$, we can always find a Hilbert space \mathcal{E} and a bounded operator $V: \mathcal{H}_2 \rightarrow \mathcal{H}_1 \otimes \mathcal{E}$, such that

$$T(A) = V^*(A \otimes \mathbb{1}_{\mathcal{E}})V \quad \forall A \in \mathcal{A}. \quad (3)$$

This follows from the fact that any normal $*$ -representation of the C^* -algebra $\mathcal{B}(\mathcal{H})$ is unitarily equivalent to the *amplification map* $A \mapsto A \otimes \mathbb{1}_{\mathcal{E}}$ for some Hilbert space \mathcal{E} (Ref. 12, Sec. 2.7). Any minimal Stinespring dilation of T that has the form (3) will be referred to as its *canonical Stinespring dilation*. The canonical Stinespring dilation is likewise unique up to unitary equivalence.

Another important structure theorem for CP maps is due to Kraus.³ It says that for any CP map $T: \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$, with \mathcal{A} being a W^* -algebra of operators on some Hilbert space \mathcal{H}' , there exists a collection of bounded operators $V_x: \mathcal{H} \rightarrow \mathcal{H}'$, such that

$$T(A) = \sum_x V_x^* A V_x, \quad (4)$$

where the series converges in the strong operator topology. If $\dim \mathcal{H} = \infty$, the set $\{V_x\}$ can be chosen in such a way that its cardinality equals the Hilbertian dimension (i.e., the cardinality of any complete orthonormal basis) of \mathcal{H} .¹¹

The Stinespring dilation (3) and the Kraus form (4) of a CP map $T: \mathcal{B}(\mathcal{H}_1) \rightarrow \mathcal{B}(\mathcal{H}_2)$ are related to one another via the correspondence

$$V\psi = \sum_x V_x \psi \otimes e_x \quad \forall \psi \in \mathcal{H}_2, \tag{5}$$

where $\{e_x\}$ is an orthonormal system in \mathcal{E} . Note that the Kraus operators $\{V_x\}$ depend on the choice of $\{e_x\}$. The adjoint operator $V^*: \mathcal{H}_1 \otimes \mathcal{E} \rightarrow \mathcal{H}_2$ acts on the elementary tensors $\psi \otimes \chi \in \mathcal{H}_1 \otimes \mathcal{E}$ as

$$V^*(\psi \otimes \chi) = \sum_x \langle e_x | \chi \rangle V_x^* \psi.$$

It is not hard to see that when \mathcal{H}_1 and \mathcal{H}_2 are both finite-dimensional, any canonical Stinespring dilation of T will give rise to at most $\dim \mathcal{H}_1 \cdot \dim \mathcal{H}_2$ Kraus operators. This is so because these Kraus operators must be linearly independent elements of the vector space $\mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ of all linear operators from \mathcal{H}_2 into \mathcal{H}_1 . Furthermore, the number of terms in such a Kraus decomposition is uniquely determined by T .¹³

3. Partial order of CP maps

The cone $\text{CP}(\mathcal{A}; \mathcal{H})$ of all normal CP maps of \mathcal{A} into $\mathcal{B}(\mathcal{H})$ can be partially ordered in the following natural fashion. Given $S, T \in \text{CP}(\mathcal{A}; \mathcal{H})$, we will write $S \leq T$ if $T - S \in \text{CP}(\mathcal{A}; \mathcal{H})$. Following Belavkin and Staszewski,⁷ we will say that S is *completely dominated by* T . Given a nonnegative real constant c , we will say that S is *completely c -dominated by* T if $S \leq cT$. Using the condition (1), we see that $S \leq T$ if and only if

$$\sum_{i,j=1}^n \langle \eta_i | S(A_i^* A_j) \eta_j \rangle \leq \sum_{i,j=1}^n \langle \eta_i | T(A_i^* A_j) \eta_j \rangle \quad \forall \eta_i \in \mathcal{H}, A_i \in \mathcal{A}; i = 1, \dots, n$$

for each $n \in \mathbb{N}$. We will use the notation $\text{CP}(\mathcal{H}_1, \mathcal{H}_2)$ (note the comma) for the set of all CP maps of $\mathcal{B}(\mathcal{H}_1)$ into $\mathcal{B}(\mathcal{H}_2)$.

B. Quantum operations

Reversible dynamics of a closed quantum-mechanical system with the Hilbert space \mathcal{H} is given, in the Schrödinger picture, by the mapping $\rho \mapsto U\rho U^*$, where ρ is a density operator on \mathcal{H} (i.e., $\text{Tr} \rho = 1$ and $\rho \geq 0$), and $U: \mathcal{H} \rightarrow \mathcal{H}$ is a unitary transformation. In the dual Heisenberg picture same dynamics is described by the mapping $A \mapsto U^* A U$ for all $A \in \mathcal{B}(\mathcal{H})$. The two descriptions are equivalent as they yield the same observed statistics, $\text{Tr}(U\rho U^* A) = \text{Tr}(\rho U^* A U)$.

On the other hand, when the system is open because it is either coupled to an environment or is being subjected to a measurement, its most general time evolution is irreversible. This is captured mathematically by means of a *quantum operation*,³ i.e., a completely positive normal linear map $T: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ with the additional constraint $T(\mathbb{1}) \leq \mathbb{1}$. In terms of the Kraus form, $T(A) = \sum_x V_x^* A V_x$, we have the bound $\sum_x V_x^* V_x \leq \mathbb{1}$. The corresponding Schrödinger-picture map on density operators, $\rho \mapsto T_*(\rho)$, is defined¹⁴ by

$$\text{Tr}[T_*(\rho)A] = \text{Tr}[\rho T(A)] \quad \forall A \in \mathcal{B}(\mathcal{H}),$$

and can then be extended to the linear span of the density operators, the trace class $\mathcal{T}(\mathcal{H})$. It follows at once that the map T_* is completely positive and trace decreasing in the sense that $\text{Tr} T_*(X) \leq \text{Tr} X$ for any $X \in \mathcal{T}(\mathcal{H})$. In order to retain proper normalization for density operators,

one usually writes the Schrödinger-picture evolution dual to T as $\rho \mapsto T_*(\rho)/\text{Tr} T_*(\rho)$. Alternatively, one says that the transformation $\rho \mapsto T_*(\rho)$ *succeeds with probability* $\text{Tr} T_*(\rho)$; this probability is equal to unity for all density operators ρ if and only if T is unital, i.e., $T(\mathbb{1}) = \mathbb{1}$, so that T_* is trace-preserving. Unital quantum operations are also referred to as *quantum channels*.¹

The Kraus theorem implies that we can write any quantum operation T as a sum of *pure operations* (Ref. 2, Sec. 2.3), i.e., maps of the form $A \mapsto X^*AX$ with $X^*X \leq \mathbb{1}$ (this is equivalent to X being a contraction, $\|X\| \leq 1$ where $\|\cdot\|$ is the usual operator norm, $\|X\| = \sup_{\psi \in \mathcal{H}} \|X\psi\|/\|\psi\|$). The qualification “pure” is usually interpreted as referring to the fact that, for any pure state $|\psi\rangle\langle\psi|$, the (un-normalized) state $X|\psi\rangle\langle\psi|X^*$ is pure as well.⁵ However, as we shall see later, it is a direct consequence of the Radon–Nikodym theorem for CP maps that T is a pure operation if and only if all operations completely dominated by it are its nonnegative multiples. This is analogous to the case of pure states on a C*-algebra \mathcal{A} : a state ω on \mathcal{A} is pure if and only if all positive linear functionals φ on \mathcal{A} , such that $\omega - \varphi$ is positive are nonnegative multiples of ω (Ref. 15, Sec. 2.3.2).

Given the canonical Stinespring dilation (3) of a quantum channel T (in which case V is an isometry), the Schrödinger-picture operation T_* can be cast in the so-called *ancilla form*

$$T_*(\rho) = \text{Tr}_{\mathcal{E}} U(\rho \otimes |\xi\rangle\langle\xi|) U^*, \tag{6}$$

where $\text{Tr}_{\mathcal{E}}(\cdot)$ denotes the partial trace over \mathcal{E} , $|\xi\rangle \in \mathcal{E}$ is a fixed unit vector, and U is the unitary extension of the partial isometry \hat{U} from $\mathcal{H}_2 \otimes [|\xi\rangle\langle\xi|]$ to $\mathcal{H}_1 \otimes \mathcal{E}$ defined by $\hat{U}(\psi \otimes \xi) = V\psi$.^{3,16} (We use $[P]$ to denote the closed subspace corresponding to the orthogonal projection P .)

Finally, note that the input and output Hilbert spaces do not have to be the same; in general, quantum operations are completely positive normal linear maps $T: \mathcal{B}(\mathcal{H}_1) \rightarrow \mathcal{B}(\mathcal{H}_2)$ with $T(\mathbb{1}_{\mathcal{H}_1}) \leq \mathbb{1}_{\mathcal{H}_2}$. The corresponding Schrödinger-picture operations are completely positive trace-decreasing maps $T_*: \mathcal{T}(\mathcal{H}_2) \rightarrow \mathcal{T}(\mathcal{H}_1)$. Most of the discussion in this section carries over to this case, modulo straightforward modifications; however, one must be careful with the ancilla representation of a general Schrödinger-picture channel T_* . The key caveat here is that the initial ancillary space and the final “traced-out” space need not be isomorphic. This yet again underscores the advantages of working in the Heisenberg picture.

C. The norm of complete boundedness

In many information-theoretic studies of noisy quantum channels one needs a quantitative measure of the “noisiness” of a channel; this is, in fact, a natural departure point for various definitions of information-carrying capacities of quantum channels.^{1,17,18} A good candidate for such a measure is the norm $\|T - \text{id}\|_?$, where the question mark refers to the fact that we have not yet specified a suitable norm.

The choice of the proper norm turns out to be a tricky matter.¹ Let \mathcal{A} and \mathcal{B} be C*-algebras, and consider a linear map $\Lambda: \mathcal{A} \rightarrow \mathcal{B}$. We cannot adopt the operator norm, defined by

$$\|\Lambda\| = \sup\{\|\Lambda(A)\| \mid A \in \mathcal{A}, \|A\| \leq 1\}, \tag{7}$$

where $\|A\|$ is the (unique) C*-norm on \mathcal{A} , because the norm $\|\Lambda \otimes \text{id}_n\|$ of the map $\Lambda \otimes \text{id}_n: \mathcal{A} \otimes \mathcal{M}_n \rightarrow \mathcal{B} \otimes \mathcal{M}_n$ can increase with n even if Λ itself is bounded (see Chap. 3 of Ref. 19). What we need is a “stabilized” version of (7). A map $\Lambda: \mathcal{A} \rightarrow \mathcal{B}$ is called *completely bounded* (CB for short) if there exists some constant $C \geq 0$ such that all the maps $\Lambda \otimes \text{id}_n: \mathcal{A} \otimes \mathcal{M}_n \rightarrow \mathcal{B} \otimes \mathcal{M}_n$ are uniformly bounded by C , i.e., $\|\Lambda \otimes \text{id}_n\| \leq C$. The CB norm $\|\Lambda\|_{\text{cb}}$ is defined to be the smallest constant C for which this holds, i.e.,

$$\|\Lambda\|_{\text{cb}} = \sup_{n \in \mathbb{N}} \|\Lambda \otimes \text{id}_n\|.$$

All CB maps have the property of “factoring through a Hilbert space,” as shown in the following key structure theorem (Theorem 3.6 in Ref. 19), given here in a slightly simplified form suitable for our needs.

Theorem II.1: (*Haagerup–Paulsen–Wittstock*) *Let \mathcal{H} and \mathcal{K} be Hilbert spaces, and let $\Lambda:\mathcal{B}(\mathcal{H})\rightarrow\mathcal{B}(\mathcal{K})$ be a CB map. Then there exist a Hilbert space \mathcal{E} and operators $V_1, V_2:\mathcal{K}\rightarrow\mathcal{H}\otimes\mathcal{E}$ with $\|V_1\|\|V_2\|\leq\|\Lambda\|_{\text{cb}}$ ($\|\cdot\|$ stands for the operator norm), such that*

$$\Lambda(A) = V_1^*(A \otimes \mathbb{1}_{\mathcal{E}})V_2. \tag{8}$$

Conversely, any map Λ of the form (8) satisfies $\|\Lambda\|_{\text{cb}}\leq\|V_1\|\|V_2\|$.

Note that the Stinespring and the Haagerup–Paulsen–Wittstock theorems together imply that any CP map is automatically CB. In fact, for a CP map T , we have $\|T\|_{\text{cb}}=\|T(\mathbb{1})\|$.²⁰ Also, the difference of two CP maps is always CB.

Theorem II.1 suggests an alternative way to define the CB norm of a map Λ , namely, as

$$\|\Lambda\|_{\text{cb}} = \inf\{\|V_1\|\|V_2\|\}, \tag{9}$$

where the infimum is taken over all possible decompositions of Λ in the form (8). Moreover, the theorem guarantees that the infimum in (9) is attained.

In quantum information theory one frequently deals with both the operation $T:\mathcal{B}(\mathcal{H})\rightarrow\mathcal{B}(\mathcal{K})$ and its (pre)dual, $T_*:\mathcal{T}(\mathcal{K})\rightarrow\mathcal{T}(\mathcal{H})$. As we mentioned in Sec. II B, T and T_* are connected by the relation $\text{Tr}[T(A)B]=\text{Tr}[AT_*(B)]$, $A\in\mathcal{B}(\mathcal{H})$, $B\in\mathcal{T}(\mathcal{K})$. This duality holds also for any normal CB map $\Lambda:\mathcal{B}(\mathcal{H})\rightarrow\mathcal{B}(\mathcal{K})$, so that when Λ is written in the form (8), we have

$$\Lambda_*(A) = \text{Tr}_{\mathcal{E}}V_2AV_1^* \quad \forall A\in\mathcal{T}(\mathcal{K}). \tag{10}$$

This motivates the definition of the dual CB norm,

$$\|\Lambda_*\|_{\text{cb}}^* = \inf\{\|V_1\|\|V_2\|\}, \tag{11}$$

where the infimum is taken over all possible decompositions of Λ_* in the form (10). It is now clear that $\|\Lambda\|_{\text{cb}}=\|\Lambda_*\|_{\text{cb}}^*$ for any normal CB map Λ , so in the future we will always write $\|\Lambda\|_{\text{cb}}$, even when working with Λ_* . In fact, the norm (11) was introduced by Kitaev²¹ under the name “diamond norm” (Kitaev used the notation $\|\Lambda\|_{\diamond}$). The equivalence of the diamond norm and the CB norm has been alluded to in the literature on quantum information theory¹⁷ but, to the best of our knowledge, no proof of the equivalence was ever presented.

The duality relation between $\Lambda:\mathcal{B}(\mathcal{H})\rightarrow\mathcal{B}(\mathcal{K})$ and $\Lambda_*:\mathcal{T}(\mathcal{K})\rightarrow\mathcal{T}(\mathcal{H})$ implies that we can also write

$$\|\Lambda\|_{\text{cb}} = \sup_{n\in\mathbb{N}}\|\Lambda_*\otimes\text{id}_n\|_1,$$

where $\|\Lambda_*\|_1 = \sup\{\|\Lambda_*(A)\|_1\mid A\in\mathcal{T}(\mathcal{K}),\|A\|_1\leq 1\}$ and $\|A\|_1 = \text{Tr}|A| \equiv \text{Tr}\sqrt{A^*A}$ is the trace norm (Sec. VI.6, Ref. 22). For this purpose we can use the well-known variational characterization of the operator norm (Theorem 3.2 in Ref. 23), namely,

$$\|A\| = \sup_{\substack{B\in\mathcal{T}(\mathcal{H}) \\ \|B\|_1\leq 1}} |\text{Tr}(AB)| \quad \forall A\in\mathcal{B}(\mathcal{H}).$$

Then for any normal CB map $\Lambda:\mathcal{B}(\mathcal{H})\rightarrow\mathcal{B}(\mathcal{K})$ we have

$$\begin{aligned} \|\Lambda\| &= \sup_{\substack{A \in \mathcal{B}(\mathcal{H}) \\ \|A\| \leq 1}} \|\Lambda(A)\| = \sup_{\substack{B \in \mathcal{T}(\mathcal{K}) \\ \|B\|_1 \leq 1}} \sup_{\substack{A \in \mathcal{B}(\mathcal{H}) \\ \|A\| \leq 1}} |\text{Tr}[\Lambda(A)B]| = \sup_{\substack{B \in \mathcal{T}(\mathcal{K}) \\ \|B\|_1 \leq 1}} \sup_{\substack{A \in \mathcal{B}(\mathcal{H}) \\ \|A\| \leq 1}} |\text{Tr}[A\Lambda_*(B)]| \\ &= \sup_{\substack{B \in \mathcal{T}(\mathcal{K}) \\ \|B\|_1 \leq 1}} \|\Lambda_*(B)\|_1 = \|\Lambda_*\|_1, \end{aligned}$$

which also implies that $\|\Lambda \otimes \text{id}_n\| = \|\Lambda_* \otimes \text{id}_n\|_1$ for all $n \in \mathbb{N}$. Taking the supremum of both sides with respect to n does the job. In a nutshell, the CB norm of a map between algebras of bounded operators on Hilbert spaces can be defined through a variational expression involving the operator norm, whereas the CB norm of the corresponding dual map between the trace classes is determined by a variational expression in the trace norm.

We now summarize the key properties of the CB norm. For any two CB maps $\Lambda: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}')$ and $\Lambda': \mathcal{B}(\mathcal{H}') \rightarrow \mathcal{B}(\mathcal{K}')$, any $A \in \mathcal{B}(\mathcal{H})$, and any $B \in \mathcal{T}(\mathcal{H}')$, we have the following:

- (1) $\|\Lambda' \circ \Lambda\|_{\text{cb}} \leq \|\Lambda'\|_{\text{cb}} \|\Lambda\|_{\text{cb}}$,
- (2) $\|\Lambda \otimes \Lambda'\|_{\text{cb}} = \|\Lambda\|_{\text{cb}} \|\Lambda'\|_{\text{cb}}$,
- (3) $\|\Lambda(A)\| \leq \|\Lambda\|_{\text{cb}} \|A\|$,
- (4) $\|\Lambda_*(B)\|_1 \leq \|\Lambda\|_{\text{cb}} \|B\|_1$.

For proofs see, e.g., the article of Kitaev²¹ or the monographs of Pisier¹⁹ and Paulsen.²⁰

III. THE RADON–NIKODYM THEOREM FOR COMPLETELY POSITIVE MAPS

In this section we review a theorem of the Radon–Nikodym type that allows for a complete classification of all CP maps S that are completely dominated by a given CP map T . As we have already mentioned, this theorem can be distilled from the more general results of Arveson⁶ and Belavkin and Staszewski.⁷ The work of Parthasarathy¹⁰ contains further developments, in particular an analogue of the Lebesgue decomposition for CP maps. The idea is to express all maps S that satisfy $S \leq T$ in the form related to the (minimal) Stinespring dilation of T ; this ‘‘Stinespring form’’ of the theorem^{6,7} is stated in Sec. III A, with the proof included in order to keep the paper self-contained. Then, in Sec. III B, we state and prove two ‘‘Kraus forms’’ of the Radon–Nikodym theorem. Finally, some general remarks are given in Sec. III C.

A. The Stinespring form

Before we state and prove the Radon–Nikodym theorem, let us recall a standard piece of notation. Given a C^* -algebra \mathcal{A} and a $*$ -homomorphism $\pi: \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H})$, the set $\{B \in \mathcal{B}(\mathcal{H}) \mid [A, B] \equiv AB - BA = 0, \forall A \in \pi(\mathcal{A})\}$ is called the *commutant* of π and is denoted by $\pi(\mathcal{A})'$.

Theorem III.1: *Consider $S, T \in \text{CP}(\mathcal{A}; \mathcal{H})$, and let (\mathcal{K}, V, π) be the minimal Stinespring dilation of T . Then $S \leq T$ if and only if there exists an operator $\hat{F} \in \pi(\mathcal{A})'$, such that $0 \leq \hat{F} \leq 1$ and*

$$S(A) = V^* \pi(A) \hat{F} V = V^* \hat{F}^{1/2} \pi(A) \hat{F}^{1/2} V$$

for all $A \in \mathcal{A}$. The operator \hat{F} is unique in the sense that if $S(A) = V^* \pi(A) Y V$ for some $Y \in \pi(\mathcal{A})'$, then $Y = \hat{F}$. We will refer to this operator \hat{F} as the *Radon–Nikodym derivative* of S with respect to T and denote it by $D_T S$.

Proof: Suppose $S \leq T$, and let (\mathcal{K}', V', π') be the minimal Stinespring dilation of S . Define an operator $\hat{G}: \mathcal{K} \rightarrow \mathcal{K}'$ by

$$\hat{G}: \pi(A) V \eta \mapsto \pi'(A) V' \eta \quad \forall A \in \mathcal{A}, \eta \in \mathcal{H},$$

and extend it to the linear span of $\pi(\mathcal{A}) V \mathcal{H}$. For any finite linear combination $\Psi = \sum_{i=1}^n \pi(A_i) V \eta_i$ we have

$$\begin{aligned} \|\hat{G}\Psi\|^2 &= \sum_{i,j=1}^n \langle \eta_i | V'^* \pi'(A_i^* A_j) V' \eta_j \rangle \\ &= \sum_{i,j=1}^n \langle \eta_i | S(A_i^* A_j) \eta_j \rangle \\ &\leq \sum_{i,j=1}^n \langle \eta_i | T(A_i^* A_j) \eta_j \rangle \\ &= \sum_{i,j=1}^n \langle \eta_i | V^* \pi(A_i^* A_j) V \eta_j \rangle = \|\Psi\|^2. \end{aligned}$$

Thus \hat{G} is a densely defined contraction, and therefore extends to a contraction from \mathcal{K} into \mathcal{K}' . We will denote this extension also by \hat{G} . For the adjoint map \hat{G}^* , we have

$$\langle \eta | V^* \hat{G}^* \pi'(A) V' \xi \rangle = \langle \hat{G} V \eta | \pi'(A) V' \xi \rangle = \langle V' \eta | \pi'(A) V' \xi \rangle = \langle \eta | V'^* \pi'(A) V' \xi \rangle = \langle \eta | S(A) \xi \rangle$$

for all $\eta, \xi \in \mathcal{H}$ and $A \in \mathcal{A}$, which implies that $V^* \hat{G}^* \pi'(A) V' \eta = S(A) \eta$.

The map \hat{G} intertwines the representations π and π' , i.e., $\hat{G} \pi(A) = \pi'(A) \hat{G}$ for any $A \in \mathcal{A}$. Indeed, for all $A, B \in \mathcal{A}$ and $\eta \in \mathcal{H}$ we have

$$\hat{G} \pi(A) \pi(B) V \eta = \hat{G} \pi(AB) V \eta = \pi'(AB) V' \eta = \pi'(A) \pi'(B) V' \eta = \pi'(A) \hat{G} \pi(B) V \eta,$$

and the desired statement follows because of the minimality of the Stinespring dilation (\mathcal{K}, V, π) . Taking adjoints, we also obtain $\pi(A) \hat{G}^* = \hat{G}^* \pi'(A)$. Letting $\hat{F} = \hat{G}^* \hat{G}$, we see that

$$\hat{F} \pi(A) = \hat{G}^* \hat{G} \pi(A) = \hat{G}^* \pi'(A) \hat{G} = \pi(A) \hat{G}^* \hat{G} = \pi(A) \hat{F},$$

which shows that $\hat{F} \in \pi(\mathcal{A})'$. Finally, for all $A \in \mathcal{A}$ and $\eta \in \mathcal{H}$ we have

$$V^* \hat{F} \pi(A) V \eta = V^* \hat{G}^* \hat{G} \pi(A) V \eta = V^* \hat{G}^* \pi'(A) V' \eta = S(A) \eta,$$

thus $S(A) = V^* \hat{F} \pi(A) V = V^* \pi(A) \hat{F} V = V^* \hat{F}^{1/2} \pi(A) \hat{F}^{1/2} V$. The uniqueness of \hat{F} follows from the minimality of (\mathcal{K}', V', π') .

The converse is clear. ■

For the special case $S, T \in \text{CP}(\mathcal{H}_1, \mathcal{H}_2)$ we can use the canonical Stinespring dilation (3) and the fact that the commutant of the algebra $\mathcal{B}(\mathcal{H}_1) \otimes \text{Cl}_{\mathcal{E}}$ is isomorphic to $\text{Cl}_{\mathcal{H}_1} \otimes \mathcal{B}(\mathcal{E})$ (Theorem IV.5.9 in Ref. 24), to deduce the following.

Corollary III.2: Let $S, T \in \text{CP}(\mathcal{H}_1, \mathcal{H}_2)$, and let $T(A) = V^*(A \otimes \mathbb{1}_{\mathcal{E}})V$ be the canonical Stinespring dilation of T . Then $S \leq T$ if and only if there exists a positive contraction $F \in \mathcal{B}(\mathcal{E})$, such that $S(A) = V^*(A \otimes F)V$ for all $A \in \mathcal{B}(\mathcal{H}_1)$.

As we already mentioned, the Radon–Nikodym theorem allows one to fully appreciate the term “pure operation.” Let \mathcal{H}_1 and \mathcal{H}_2 be Hilbert spaces, and consider the map $T(A) = X^*AX$, where $X: \mathcal{H}_2 \rightarrow \mathcal{H}_1$ is a contraction. Clearly, X^*AX is the canonical Stinespring dilation of T so, by Theorem III.1, any $S \in \text{CP}(\mathcal{H}_1, \mathcal{H}_2)$ that satisfies $S \leq T$ must be of the form λX^*AX for some $\lambda \in [0, 1]$.

Theorem III.1 can also be used to characterize completely all ways to write a given $T \in \text{CP}(\mathcal{A}; \mathcal{H})$ as a finite sum $\sum_i T_i$, with $T_i \in \text{CP}(\mathcal{A}; \mathcal{H})$ for all i . It is actually the resulting theorem, stated below, that is referred to as the “Radon–Nikodym theorem for CP maps” in the quantum information literature.¹⁸

Theorem III.3: Consider a map $T \in \text{CP}(\mathcal{A}; \mathcal{H})$ with the canonical Stinespring dilation (\mathcal{K}, V, π) . For any finite decomposition $T = \sum_i T_i$ with $T_i \in \text{CP}(\mathcal{A}; \mathcal{H})$ there exist unique positive operators $\hat{F}_i \in \pi(\mathcal{A})'$ that satisfy $\sum_i \hat{F}_i = \mathbb{1}_{\mathcal{K}}$, such that $T_i(A) = V^* \pi(A) \hat{F}_i V$.

Proof: Apply Theorem III.1 separately to each pair (T_i, T) , and let $\hat{F}_i = D_T T_i$. Then $T(A) = \sum_i V^* \pi(A) \hat{F}_i V = V^* \pi(A) V$, and $\sum_i \hat{F}_i = \mathbb{1}_{\mathcal{K}}$ by the uniqueness part of Theorem III.1. ■

Remark: The decomposition $T = \sum_i T_i$ is a particularly simple instance of a CP instrument.² As such, it is not difficult to extract Theorem III.3 from more general results of Ozawa.⁹ □

B. The Kraus form

Theorem III.1 can be restated in a simple way in terms of the Kraus form of a CP map. In order to do this, we need some additional machinery (Sec. II.15 in Ref. 25).

Let X be a set. Any function $K: X \times X \rightarrow \mathbb{C}$ is called a *kernel on X* . The set $\mathcal{K}(X)$ of all kernels on X is a vector space, with the corresponding algebraic operations defined pointwise on $X \times X$. We say that a kernel $K \in \mathcal{K}(X)$ is *positive-definite*, and write $K \geq 0$, if for each $n \in \mathbb{N}$ we have

$$\sum_{i,j=1}^n \overline{c_i c_j} K(x_i, x_j) \geq 0 \quad \forall x_i \in X, c_i \in \mathbb{C}; \quad i = 1, \dots, n.$$

Given a pair of kernels $K, K' \in \mathcal{K}(X)$, we will write $K \leq K'$ if $K' - K$ is positive-definite. Note that a positive-definite kernel is automatically Hermitian, i.e., $K(x, y) = \overline{K(y, x)}$.

According to the fundamental theorem of Kolmogorov, for any positive-definite kernel $K \in \mathcal{K}(X)$ there exist a Hilbert space \mathcal{H}_K and a map $v_K: X \rightarrow \mathcal{H}_K$ such that $\langle v_K(x) | v_K(y) \rangle = K(x, y)$ for all $x, y \in X$, and the set $\{v_K(x) | x \in X\}$ is total in \mathcal{H}_K . The pair (\mathcal{H}_K, v_K) is referred to as the *Kolmogorov decomposition* of \mathcal{K} and is unique up to unitary equivalence.

After these preparations, we may state our first result.

Theorem III.4: *Consider two maps $S, T \in \text{CP}(\mathcal{H}_1, \mathcal{H}_2)$. Let $\{V_x\}_{x \in X}$ be a Kraus decomposition of T induced by the canonical Stinespring dilation $T(A) = V^*(A \otimes \mathbb{1}_{\mathcal{E}})V$, as prescribed in (5). Then $S \leq T$ if and only if*

$$S(A) = \sum_{x,y \in X} K(x,y) V_x^* A V_y$$

for some positive-definite kernel $K \in \mathcal{K}(X)$ with $K \leq I$, where I is the Kronecker kernel $I(x, y) \equiv \delta_{xy}$.

Proof: Suppose $S \leq T$. By Corollary III.2, $S(A) = V^*(A \otimes F)V$ for some positive contraction $F \in \mathcal{B}(\mathcal{E})$. Let $\{e_x\}_{x \in X}$ be the orthonormal system in \mathcal{E} , determined by V and $\{V_x\}$ from (5). Then for any $\eta \in \mathcal{H}_2$ we have

$$S(A)\eta = V^*(A \otimes F)V\eta = V^* \left(\sum_{y \in X} A V_y \eta \otimes F e_y \right) = \sum_{x,y \in X} \langle e_x | F e_y \rangle V_x^* A V_y \eta.$$

Define the kernel $K \in \mathcal{K}(X)$ by setting $K(x, y) := \langle e_x | F e_y \rangle$. Then $0 \leq F \leq \mathbb{1}$ implies that $0 \leq K \leq I$.

Conversely, suppose we are given

$$T(A) = \sum_{x \in X} V_x^* A V_x$$

and

$$S(A) = \sum_{x,y \in X} K(x,y) V_x^* A V_y$$

for some $K \in \mathcal{K}(X)$ such that $0 \leq K \leq I$. Let (\mathcal{H}_K, v_K) be the Kolmogorov decomposition of K , and let $\text{fin}(X)$ be the set of all finite subsets of X . Define an operator $G: \mathcal{E} \rightarrow \mathcal{H}_K$ by

$$G: \sum_{x \in X_0} c_x e_x \mapsto \sum_{x \in X_0} c_x v_K(x) \quad \forall c_x \in \mathbb{C}, X_0 \in \text{fin}(X).$$

It is easy to see that, for any $X_0 \in \text{fin}(X)$,

$$\left\| \sum_{x \in X_0} c_x e_x \right\|^2 = \sum_{x \in X_0} |c_x|^2 = 0$$

implies $c_x = 0$ for all $x \in X_0$, and consequently

$$\left\| G \left(\sum_{x \in X_0} c_x e_x \right) \right\|^2 = \sum_{x,y \in X_0} \overline{c_x} c_y K(x,y) \leq \sum_{x \in X_0} |c_x|^2 = 0,$$

where the last equality above follows because $K \leq I$. Thus G extends to a well-defined linear operator on \mathcal{E} , which we will also denote by G . Let $F = G^*G$. Then $\langle e_x | F e_y \rangle = \langle v_K(x) | v_K(y) \rangle = K(x,y)$, and $0 \leq K \leq I$ implies that $0 \leq F \leq \mathbb{1}_{\mathcal{E}}$. Thus, for all $A \in \mathcal{B}(\mathcal{H}_1)$ and $\eta \in \mathcal{H}_2$ we have

$$S(A)\eta = \sum_{x,y} K(x,y) V_x^* A V_y = \sum_{x,y \in X} \langle e_x | F e_y \rangle V_x^* A V_y = V^*(A \otimes F)V,$$

so that $S \leq T$ by Corollary III.2. ■

Remarks: (1) When the set $\{V_x\}$ is finite, Theorem III.4 says that $S \leq T$ for $T(A) = \sum_x V_x^* A V_x$ if and only if $S(A) = \sum_{x,y} M_{xy} V_x^* A V_y$ for some matrix $M = [M_{xy}]$ with $0 \leq M \leq 1$.

(2) Since we deal only with separable Hilbert spaces, the index set X is at most countably infinite. □

Another Kraus form of the Radon–Nikodym theorem can be proved directly, without recourse to the theory of positive-definite kernels.

Theorem III.5: *Consider two maps $S, T \in \text{CP}(\mathcal{H}_1, \mathcal{H}_2)$. Then $S \leq T$ if and only if there exist a Kraus decomposition $T(A) = \sum_x W_x^* A W_x$, induced by the canonical Stinespring dilation of T , and a set $\{\lambda_x | \lambda_x \in [0,1]\}$, such that $S(A) = \sum_x \lambda_x W_x^* A W_x$.*

Proof: Suppose $S \leq T$. Let $T(A) = V^*(A \otimes \mathbb{1}_{\mathcal{E}})V$ be the canonical Stinespring dilation of T . Then Corollary III.2 says that $S(A) = V^*(A \otimes F)V$ for some positive contraction $F \in \mathcal{B}(\mathcal{E})$. Write down the spectral decomposition $F = \sum_x \lambda_x |\phi_x\rangle\langle\phi_x|$, so that $\lambda_x \in [0,1]$ and $\langle\phi_x | \phi_y\rangle = \delta_{xy}$. Let $\{W_x\}$ be the Kraus decomposition of T determined from (5) by V and $\{\phi_x\}$. Then for any $\eta \in \mathcal{H}_2$ we have

$$S(A)\eta = V^*(A \otimes F)V\eta = V^* \left(\sum_y \lambda_y A W_y \eta \otimes \phi_y \right) = \sum_{x,y} \lambda_y \langle\phi_x | \phi_y\rangle W_x^* A W_y \eta = \sum_x \lambda_x W_x^* A W_x.$$

The converse follows readily from the fact that the map $A \mapsto \sum_x (1 - \lambda_x) W_x^* A W_x$ is CP for any choice of $\{W_x\}$ and $\{\lambda_x\}$ with $\lambda_x \in [0,1]$. ■

C. General remarks

Before we go on, we would like to pause and make some general comments about the significance of the Radon–Nikodym theorem for CP maps at large.

The real power of this theorem lies in the fact that it contains the “traditional” forms of the Radon–Nikodym theorem as special cases. In order to see this, we will need the following result (see Corollary IV.3.5 and Proposition IV.3.9 in Ref. 24): a positive map T from a C^* -algebra \mathcal{A} to another C^* -algebra \mathcal{B} is automatically completely positive whenever at least one of \mathcal{A} and \mathcal{B} is Abelian.

With this in mind, let us observe that any positive linear functional φ on a C^* -algebra \mathcal{A} is a positive map from \mathcal{A} to \mathbb{C} , and therefore is CP. When we apply the Stinespring theorem to φ , we simply recover the GNS representations $(\mathcal{H}, \pi, \Omega)$ of \mathcal{A} induced by φ , where \mathcal{H} is the Hilbert space

of the representation, π is a $*$ -isomorphism between \mathcal{A} and a suitable C^* -subalgebra of $\mathcal{B}(\mathcal{H})$, and $\Omega \in \mathcal{H}$ is cyclic for π , i.e., $\mathcal{H} = \pi(\mathcal{A})\Omega$. Of course, we have then $\varphi(A) = \langle \Omega | \pi(A)\Omega \rangle$ for all $A \in \mathcal{A}$.

Consider first the Abelian case. Let X be a compact Hausdorff space, and let \mathcal{A} be the commutative C^* -algebra $\mathcal{C}(X)$ of all complex-valued continuous functions on X . Let φ be a positive linear functional on $\mathcal{C}(X)$. By the Riesz–Markov theorem [see Theorem IV.14 (Ref. 22)], there exists a unique Baire measure μ on X such that $\varphi(f) = \int_X f(x) d\mu(x)$, $\forall f \in \mathcal{C}(X)$. If φ is a state [i.e., $\varphi(\mathbb{1}_X) = 1$ where $\mathbb{1}_X$ is, of course, the function on X that is identically equal to 1], then μ is a probability measure. The GNS construction yields the cyclic representation $(\mathcal{H}, \pi, \Omega)$, where $\mathcal{H} = \mathcal{L}^2(X, d\mu)$, $[\pi(f)g](x) = f(x)g(x)$, and $\Omega = \mathbb{1}_X$, such that

$$\varphi(f) = \langle \Omega | \pi(f)\Omega \rangle = \int_X f(x) d\mu(x).$$

This is the minimal Stinespring dilation of the CP map $\varphi: \mathcal{C}(X) \rightarrow \mathbb{C}$; more precisely, we have the isometry $V: \mathbb{C} \rightarrow \mathcal{L}^2(X, d\mu)$ defined by $Vc = c\Omega$, so that $\varphi(f) = V^* \pi(f) V$. Now suppose we are given another positive linear functional η on $\mathcal{C}(X)$ such that $\eta \leq \varphi$, i.e., $\eta(f) \leq \varphi(f)$ for every non-negative $f \in \mathcal{C}(X)$. Then Theorem III.1 states that there exists a non-negative function $\rho \in \pi(\mathcal{C}(X))' \subseteq \mathcal{L}^\infty(X, d\mu)$ such that $\eta(f) = V^* \pi(f) \rho V$, i.e.,

$$\eta(f) = \langle \Omega | \rho \pi(f)\Omega \rangle = \int_X \rho(x) f(x) d\mu(x).$$

Again, by the Riesz–Markov theorem, there exists a unique Baire measure ν on X such that $\eta(f) = \int_X f(x) d\nu(x)$. It is easy to see that the function ρ is precisely the measure-theoretic Radon–Nikodym derivative $d\nu/d\mu$.

The noncommutative case is dealt with in a similar manner. Namely, if φ is a state on a unital C^* -algebra \mathcal{A} that admits the cyclic representation $(\mathcal{H}, \pi, \Omega)$, then any positive linear functional η on \mathcal{A} such that $\eta \leq \varphi$ has the form $\eta(A) = \langle \Omega | \pi(A) F \Omega \rangle$ for a unique positive contraction $F \in \pi(\mathcal{A})'$. This is, of course, the familiar Radon–Nikodym theorem for states on C^* -algebras (see Theorem 2.3.19 in Ref. 15).

IV. PARTIAL ORDERING OF QUANTUM OPERATIONS

The first series of problems we tackle by means of the Radon–Nikodym theorems of Sec. III is connected to the partial ordering of quantum operations with respect to the relation of complete domination, defined in Sec. II.

As mentioned already, all quantum operations $T: \mathcal{B}(\mathcal{H}_1) \rightarrow \mathcal{B}(\mathcal{H}_2)$ must satisfy $T(\mathbb{1}_{\mathcal{H}_1}) \leq \mathbb{1}_{\mathcal{H}_2}$. It turns out that this normalization condition imposes severe restrictions on the structure of their order intervals. In particular, as shown in the following Proposition, no nontrivial difference of quantum channels can be a CP map.

Proposition IV.1: *Let $S, T \in \text{CP}(\mathcal{H}_1, \mathcal{H}_2)$ be quantum channels. Then $T - S \in \text{CP}(\mathcal{H}_1, \mathcal{H}_2)$ if and only if $S = T$.*

Proof: Suppose $T - S \in \text{CP}(\mathcal{H}_1, \mathcal{H}_2)$, or, equivalently, $S \leq T$. Then Theorem III.5 implies that there exists a Kraus decomposition $T(A) = \sum_x W_x^* A W_x$ such that $S(A) = \sum_x \lambda_x W_x^* A W_x$ with $0 \leq \lambda_x \leq 1$. Because both S and T are channels, $S(\mathbb{1}) = T(\mathbb{1}) = \mathbb{1}$, which implies that $\sum_x (1 - \lambda_x) W_x^* W_x = 0$. Since each term in this sum is a positive operator, the only possibility is that $\lambda_x = 1$ for all x , or $S = T$. The converse is obvious. ■

Remark: To obtain an even simpler proof of this proposition, we can use the fact that, for a CP map T , $\|T\|_{\text{cb}} = \|T(\mathbb{1})\|$ (cf. Sec. II C). Indeed, if S and T are channels, then $T(\mathbb{1}) = S(\mathbb{1}) = \mathbb{1}$, and the assumption that $T - S$ is CP yields $\|T - S\|_{\text{cb}} = \|T(\mathbb{1}) - S(\mathbb{1})\| = 0$, or $S = T$. In fact, the same method shows that if S and T are two CP maps with $S(\mathbb{1}) = T(\mathbb{1})$, then $S - T$ cannot be a CP map. □

The only possible order relation between a pair of quantum channels S and T is that, say, T completely c -dominates S for some $c > 1$. The latter condition follows from Proposition IV.1 and from the fact that $S \leq cT$ implies $1 \leq c1$, which is (trivially) possible only if $c \geq 1$. In fact, as pointed out by Parthasarathy,¹⁰ there are pairs of channels T, T' for which there exist constants $c, c' > 1$ such that $T' \leq cT$ and $T \leq c'T'$. To show this, let S_1 and S_2 be arbitrary channels, and define $T = \lambda S_1 + (1 - \lambda)S_2$ and $T' = \lambda' S_1 + (1 - \lambda')S_2$, where $0 < \lambda, \lambda' < 1$. Then, setting $c = [\lambda(1 - \lambda)]^{-1}$ and $c' = [\lambda'(1 - \lambda')]^{-1}$, we see that indeed $T' \leq cT$ and $T \leq c'T'$. In Parthasarathy's terminology,¹⁰ T and T' are *uniformly equivalent*; this is written $T \equiv_u T'$, and is an equivalence relation.

The next problem we consider has to do with an alternative way to (partially) order quantum operations by means of orthogonal projections on a suitably enlarged Hilbert space. To this end we need to recall some facts about the so-called *positive operator-valued measures* (POVM's for short) (Sec. 3.1 in Ref. 2). Let X be a topological space, Σ_X the σ -algebra of all Borel subsets of X , and \mathcal{H} a Hilbert space. A map $M: \Sigma_X \rightarrow \mathcal{B}(\mathcal{H})$ is a POVM on (the Borel subsets of) X if it has the following properties:

- (1) (normalization) $M(\emptyset) = 0$ and $M(X) = 1$,
- (2) (positivity) $M(\Delta) \geq 0$ for all $\Delta \in \Sigma_X$,
- (3) (σ -additivity) if $\{\Delta_i\}$ is a countable collection of pairwise disjoint Borel sets in X , then $M(\cup_i \Delta_i) = \sum_i M(\Delta_i)$, where the sum converges in the weak operator topology.

A POVM that satisfies an additional requirement that each $M(\Delta)$ is an orthogonal projection, i.e., $M(\Delta)^2 = M(\Delta)$, is called a *projection-valued measure* (PVM). The resulting resolution of identity is an orthogonal one. The celebrated Naimark dilation theorem (see Theorem 9.3.2 in Ref. 2) says that for every POVM $M: \Sigma_X \rightarrow \mathcal{B}(\mathcal{H})$ there exist a Hilbert space \mathcal{K} , a unitary $U: \mathcal{H} \rightarrow \mathcal{K}$, a Hilbert space $\tilde{\mathcal{K}}$ containing \mathcal{K} as a closed subspace, and a PVM $E: \Sigma_X \rightarrow \mathcal{B}(\tilde{\mathcal{K}})$, such that, for any $\Delta \in \Sigma_X$, $M(\Delta) = U^* P E(\Delta) P U$, where P is the orthogonal projection from $\tilde{\mathcal{K}}$ onto \mathcal{K} . Furthermore, we can define the partial isometry $V: \mathcal{H} \rightarrow \tilde{\mathcal{K}}$ (with the final projection P) by $V = P U$, so that $M(\Delta) = V^* E(\Delta) V$.²⁶

With these lengthy preliminaries out of the way, we can proceed to state and prove our result.

Theorem IV.2: *Consider quantum operations $T_i \in \text{CP}(\mathcal{H}_1, \mathcal{H}_2)$, $i = 1, \dots, n$, that satisfy $T_1 \leq T_2 \leq \dots \leq T_n$. Then there exist a Hilbert space \mathcal{H} , an isometry $V: \mathcal{H}_2 \rightarrow \mathcal{H}_1 \otimes \mathcal{H}$, and orthogonal projections $\Pi_i \in \mathcal{B}(\mathcal{H})$ such that*

- (1) $T_i(A) = V^*(A \otimes \Pi_i)V$, $1 \leq i \leq n$,
- (2) $\Pi_1 \leq \Pi_2 \leq \dots \leq \Pi_n$.

Conversely, if items 1 and 2 above hold for quantum operations $T_i \in \text{CP}(\mathcal{H}_1, \mathcal{H}_2)$ with some \mathcal{H} , V , and $\{\Pi_i\}$, then $T_1 \leq T_2 \leq \dots \leq T_n$.

Proof: Suppose that $\{T_i\}$ satisfy the hypothesis of the theorem. Without loss of generality we may take T_n to be a channel, for if not, then we can append to $\{T_i\}_{i=1}^n$ the channel $T_{n+1}(A) = M^* A M + T_n(A)$, where $M: \mathcal{H}_2 \rightarrow \mathcal{H}_1$ is an operator defined, up to a unitary, through $M^* M = 1 - T_n(1)$, so that the resulting collection $\{T_i\}_{i=1}^{n+1}$ still satisfies $T_1 \leq T_2 \leq \dots \leq T_{n+1}$.

Define quantum operations S_i , $i = 1, \dots, n$, by $S_1 = T_1$ and $S_i = T_i - T_{i-1}$, $1 < i \leq n$. Then $T_k = \sum_{i=1}^k S_i$, $1 \leq k \leq n$. If $T_n(A) = W^*(A \otimes 1_{\mathcal{E}})W$ is the canonical Stinespring dilation of T_n , Theorem III.3 states that there exist positive operators $F_i \in \mathcal{B}(\mathcal{E})$ such that $S_i(A) = W^*(A \otimes F_i)W$, and $\sum_i F_i = 1_{\mathcal{E}}$. By the Naimark dilation theorem there exist a Hilbert space \mathcal{H} , an isometry $\tilde{V}: \mathcal{E} \rightarrow \mathcal{H}$, and a PVM $\{E_i\}_{i=1}^n$, $E_i \in \mathcal{B}(\mathcal{H})$, such that $F_i = \tilde{V}^* E_i \tilde{V}$, $1 \leq i \leq n$. Thus we can write $S_i(A) = V^*(A \otimes E_i)V$, where the isometry $V: \mathcal{H}_2 \rightarrow \mathcal{H}_1 \otimes \mathcal{H}$ is defined by $V = (1_{\mathcal{H}_1} \otimes \tilde{V})W$.

For each k , $1 \leq k \leq n$, let $\Pi_k = \sum_{i=1}^k E_i$. Since $\{E_i\}$ is an orthogonal resolution of identity, each Π_k is an orthogonal projection, and $\Pi_k \leq \Pi_l$ for $k \leq l$ by construction. Furthermore,

$$T_k(A) = \sum_{i=1}^k S_i(A) = \sum_{i=1}^k V^*(A \otimes E_i)V = V^*(A \otimes \Pi_k)V, \quad 1 \leq k \leq n,$$

and the forward direction is proved. The proof of the reverse direction is straightforward. ■

It is pertinent to remark that there are situations when the correspondence between POVM's with values in a suitable Hilbert space and decompositions of a given quantum channel T into completely positive summands is not merely a nice mathematical device, but in fact acquires direct physical significance. For instance, Gregoratti and Werner²⁷ have exploited this correspondence in a scheme for recovery of classical and quantum information from noise by making a generalized quantum measurement (described by a POVM²⁸) on the “environment” Hilbert space of a noisy quantum channel [the Hilbert space \mathcal{E} in the “ancilla” form (6)].

V. CHARACTERIZATION OF QUANTUM OPERATIONS BY POSITIVE OPERATORS

The correspondence between linear maps from a matrix algebra \mathcal{M}_m into a matrix algebra \mathcal{M}_n and linear functionals on $\mathcal{M}_n \otimes \mathcal{M}_m$ (or, by the Riesz lemma, linear operators on $\mathbb{C}_n \otimes \mathbb{C}_m$) has been treated extensively in a variety of forms in the mathematical literature (see, e.g., Refs. 13, 29–32 for a sampling of results related to positive and completely positive maps). More recently, this correspondence has been exploited fruitfully in some quantum information-theoretic contexts, such as optimal cloning maps,³³ optimal teleportation protocols,³⁴ separability criteria for entangled states,³⁵ or entanglement generation.^{36,37} In this section we will show that the one-to-one correspondence between positive operators on $\mathbb{C}_n \otimes \mathbb{C}_m$ and CP maps $T: \mathcal{M}_m \rightarrow \mathcal{M}_n$ (known as the “Jamiolkowski isomorphism” in the quantum information community) can be derived using the Radon–Nikodym machinery. We also comment on how this can be accomplished in the infinite-dimensional case with unbounded operators.

A. The Jamiolkowski isomorphism

In this section we consider quantum operations $T: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$ in the case of $\dim \mathcal{H} = m < \infty$ and $\dim \mathcal{K} = n < \infty$. Let $\{e_i\}_{i=1}^m$ and $\{f_\mu\}_{\mu=1}^n$ be fixed orthonormal bases of \mathcal{H} and \mathcal{K} . (We will use italic indices for the “input” Hilbert space, and greek ones for the “output” Hilbert space.) Let τ be the tracial state on \mathcal{M}_m , $\tau(A) = m^{-1} \text{Tr} A$, and consider the channel $\Phi(A) := \tau(A) \mathbb{1}_{\mathcal{K}}$. It is convenient to write Φ in the Kraus form

$$\Phi(A) := \sum_{i=1}^m \sum_{\mu=1}^n V_{i\mu}^* A V_{i\mu},$$

where $V_{i\mu} = (1/\sqrt{m})|e_i\rangle\langle f_\mu|$. Note that these mn Kraus operators are linearly independent, which agrees with the minimality requirement. Setting $\mathcal{E} = \mathcal{K} \otimes \mathcal{H}$, we obtain the canonical Stinespring dilation $\Phi(A) = V_\Phi^*(A \otimes \mathbb{1}_{\mathcal{E}})V_\Phi$, where

$$V_\Phi \psi = \sum_{i=1}^m \sum_{\mu=1}^n V_{i\mu} \psi \otimes f_\mu \otimes e_i.$$

Whenever we need to specify the dimensions m and n explicitly, we will write $\Phi_{m,n}$ instead of Φ , $V_{m,n}$ instead of V_Φ , etc.

We must emphasize again that the main result of this section, stated as Theorem V.1 below, is not new. Indeed, it has appeared in numerous papers on quantum information theory.^{33–37} Our contribution here is to present a new proof of this result that clearly exhibits the Jamiolkowski isomorphism in the Radon–Nikodym framework.

Theorem V.1: *In the notation described above, any CP map $T: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$ is completely m^2 -dominated by Φ . There exists a unique operator $F_T \in \mathcal{B}(\mathcal{E})$ with $0 \leq F_T \leq m^2 \mathbb{1}_{\mathcal{E}}$, such that $\mathcal{D}_\Phi T = \mathbb{1}_{\mathcal{H}} \otimes F_T$, i.e., $T(A) = V_\Phi^*(A \otimes F_T)V_\Phi$. The action of T on any $A \in \mathcal{B}(\mathcal{H})$ can also be expressed in terms of F_T only, namely, as*

$$T(A) = \frac{1}{m} \text{Tr}_{\mathcal{H}}[(\mathbb{1}_{\mathcal{K}} \otimes A^\top)F_T], \quad (12)$$

where A^\top denotes the matrix transpose of A in the basis $\{e_i\}$. Furthermore, T is a quantum operation if and only if $\text{Tr}_{\mathcal{H}} F_T \leq m \mathbb{1}_{\mathcal{K}}$.

Proof: Define $\Psi = (1/\sqrt{m}) \sum_{i=1}^m e_i \otimes e_i$, and let $H_T = T \otimes \text{id}(|\Psi\rangle\langle\Psi|)$. The matrix elements of H_T are given explicitly by

$$\langle f_\mu \otimes e_i | H_T (f_\nu \otimes e_j) \rangle = \frac{1}{m} \langle f_\mu | T(|e_i\rangle\langle e_j|) f_\nu \rangle.$$

For all $A \in \mathcal{B}(\mathcal{H})$ and $\psi \in \mathcal{K}$ we have

$$\begin{aligned} V_\Phi^*(A \otimes H_T) V_\Phi \psi &= \frac{1}{m} \sum_{i,j=1}^m \sum_{\mu,\nu=1}^n \langle e_i | A e_j \rangle \langle f_\mu \otimes e_i | H_T (f_\nu \otimes e_j) \rangle |f_\mu\rangle \langle f_\nu | \psi \rangle \\ &= \frac{1}{m^2} \sum_{\mu,\nu=1}^n \langle e_i | A e_j \rangle \langle f_\mu | T(|e_i\rangle\langle e_j|) f_\nu \rangle |f_\mu\rangle \langle f_\nu | \psi \rangle \\ &\equiv \frac{1}{m^2} T(A) \psi, \end{aligned}$$

so that $T \leq m^2 \Phi$ and $\mathbb{1}_{\mathcal{H}} \otimes m^2 H_T = D_\Phi T$ by Corollary III.2. Let $F_T = m^2 H_T$. From the uniqueness of the Radon–Nikodym derivative $D_\Phi T$ it follows that F_T determines T uniquely.

To prove Eq. (12), we need the following useful identity.

Lemma V.2: For all $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{E})$, we have

$$V_\Phi^*(A \otimes B) V_\Phi = \frac{1}{m} \text{Tr}_{\mathcal{H}}[(\mathbb{1}_{\mathcal{K}} \otimes A^\top) B].$$

Proof: Proceed by direct computation; for an arbitrary $\psi \in \mathcal{K}$, we have

$$\begin{aligned} \text{Tr}_{\mathcal{H}}[(\mathbb{1}_{\mathcal{K}} \otimes A^\top) B] \psi &= \left(\text{Tr}_{\mathcal{H}} \sum_{i,j,k=1}^m \sum_{\mu,\nu=1}^n \langle e_j | A e_i \rangle \langle f_\mu \otimes e_j | B (f_\nu \otimes e_k) \rangle |f_\mu\rangle \langle f_\nu | \otimes |e_i\rangle \langle e_k| \right) \psi \\ &= \sum_{i,j=1}^m \sum_{\mu,\nu=1}^n \langle e_j | A e_i \rangle \langle f_\mu \otimes e_j | B (f_\nu \otimes e_i) \rangle |f_\mu\rangle \langle f_\nu | \psi \rangle \\ &\equiv m V_\Phi^*(A \otimes B) V_\Phi \psi, \end{aligned}$$

and the lemma is proved. ■

This establishes Eq. (12). Finally, if T is a quantum operation, then $T(\mathbb{1}_{\mathcal{H}}) \leq \mathbb{1}_{\mathcal{K}}$. From Lemma V.2 it follows that $T(\mathbb{1}_{\mathcal{H}}) = (1/m) \text{Tr}_{\mathcal{H}} F_T$, that is, $\text{Tr}_{\mathcal{H}} F_T \leq m \mathbb{1}$. Conversely, if $T(\mathbb{1}_{\mathcal{H}}) = V_\Phi^*(\mathbb{1}_{\mathcal{K}} \otimes F_T) V_\Phi \leq \mathbb{1}_{\mathcal{K}}$, we have $\text{Tr}_{\mathcal{H}} F_T \leq m \mathbb{1}_{\mathcal{K}}$ by Lemma V.2. The theorem is proved. ■

Let $T(A) = V^*(A \otimes \mathbb{1}_{\mathcal{F}}) V$ be the canonical Stinespring dilation of T . Then it is easily shown that $\dim \mathcal{H} \cdot \dim \mathcal{F} = \text{rank } D_\Phi T$, that is $\dim \mathcal{F} = \text{rank } F_T$. Indeed, Theorems III.5 and V.1 together imply that for any CP map $T: \mathcal{M}_m \rightarrow \mathcal{M}_n$ there exist operators $\{K_i^T\}_{i=1}^N$ from \mathcal{M}_n into \mathcal{M}_m , such that $\Phi_{m,n}(A) = \sum_{i=1}^N (K_i^T)^* A K_i^T$ and $T(A) = \sum_{i=1}^N \lambda_i (K_i^T)^* A K_i^T$, where $\{\lambda_i\}$ are the (non-negative) eigenvalues of F_T . The Kraus operators $\{K_i^T\}_{i=1}^N$ are linearly independent, and are determined by the isometry V_Φ and the eigenvectors $\{\xi_i\}_{i=1}^N$ of F_T through $V_\Phi \psi = \sum_{i=1}^N V_i^T \psi \otimes \xi_i$. Therefore $N \equiv mn$. The number of nonzero terms in the corresponding Kraus decomposition of T is equal to $\text{rank } F_T$, so that $\dim \mathcal{F} = \text{rank } F_T$.

Last we would like to show how the Radon–Nikodym derivative $D_\Phi T$ transforms under composition of CP maps. Consider two CP maps $T_1: \mathcal{M}_m \rightarrow \mathcal{M}_n$ and $T_2: \mathcal{M}_n \rightarrow \mathcal{M}_d$. According to Theorem V.1 we can write

$$T_1(A) = V_{m,n}^*(A \otimes F_1)V_{m,n}, \quad T_2(B) = V_{n,d}^*(B \otimes F_2)V_{n,d}$$

for uniquely determined positive operators $F_1 \in \mathcal{M}_n \otimes \mathcal{M}_m$ and F_2 on $\mathcal{M}_d \otimes \mathcal{M}_n$. For any $A \in \mathcal{M}_m$, we have

$$\begin{aligned} T_2 \circ T_1(A) &= V_{n,d}^*(T_1(A) \otimes F_2)V_{n,d} \\ &= V_{n,d}^*(V_{m,n}^*(A \otimes F_1)V_{m,n} \otimes F_2)V_{n,d} \\ &= V_{n,d}^*(V_{m,n}^* \otimes \mathbb{1}_{d \times n})(A \otimes F_1 \otimes F_2)(V_{m,n} \otimes \mathbb{1}_{d \times n})V_{n,d}, \end{aligned}$$

where $\mathbb{1}_{d \times n}$ denotes the identity operator on the dilation space $\mathbb{C}^d \otimes \mathbb{C}^n$ of T_2 . Let $\{e_i\}_{i=1}^m$, $\{f_\mu\}_{\mu=1}^n$, and $\{\phi_x\}_{x=1}^d$ be orthonormal bases of \mathbb{C}^m , \mathbb{C}^n , and \mathbb{C}^d , respectively. Then for any $A \in \mathcal{M}_m$ and any $\psi \in \mathbb{C}^d$ we have

$$\begin{aligned} T_2 \circ T_1(A)\psi &= \frac{1}{mn} \sum_{i,j=1}^m \sum_{\mu,\nu=1}^n \sum_{x,y=1}^d \langle e_i | A e_j \rangle \langle \phi_x | \psi \rangle \langle f_\mu \otimes e_i | F_1(f_\nu \otimes e_j) \rangle \langle \phi_x \otimes f_\mu | F_2(\phi_y \otimes f_\nu) \rangle \\ &= \frac{1}{m} \sum_{i,j=1}^m \sum_{x,y=1}^d \left(\frac{1}{n} \sum_{\mu,\nu=1}^n \langle \phi_x \otimes f_\mu | F_2(\phi_y \otimes f_\nu) \rangle \langle f_\mu \otimes e_i | F_1(f_\nu \otimes e_j) \rangle \right) \langle e_i | A e_j \rangle \langle \phi_x | \psi \rangle. \end{aligned}$$

Let $\Omega = (1/\sqrt{n}) \sum_{\mu=1}^n f_\mu \otimes f_\mu$. Define an operator F_{21} on $\mathbb{C}^d \otimes \mathbb{C}^m$ by

$$\langle \phi_x \otimes e_i | F_{21}(\phi_y \otimes e_j) \rangle = \langle \phi_x \otimes \Omega \otimes e_i | (F_2 \otimes F_1)(\phi_y \otimes \Omega \otimes e_j) \rangle.$$

Then it is evident from the calculations above that we can write $T_2 \circ T_1(A) = V_{m,d}^*(A \otimes F_{21})V_{m,d}$. By the uniqueness of the Radon–Nikodym derivative, $\mathbb{1}_m \otimes F_{21} = \mathbf{D}_{\Phi_{m,d}}(T_2 \circ T_1)$. Defining the conditional expectation M_Ω from $\mathcal{M}_d \otimes \mathcal{M}_n^{\otimes 2} \otimes \mathcal{M}_m$ onto $\mathcal{M}_d \otimes \mathcal{M}_m$ by

$$M_\Omega(A \otimes B \otimes C) = \langle \Omega | B \Omega \rangle (A \otimes C) \quad \forall A \in \mathcal{M}_d, B \in \mathcal{M}_n^{\otimes 2}, C \in \mathcal{M}_m,$$

we can write more succinctly $F_{21} = M_\Omega(F_2 \otimes F_1)$.

B. Generalization to arbitrary faithful states

The construction described in Sec. V A also goes through if, instead of the tracial state τ , we take an arbitrary *faithful* state ω . As is well-known, for any such state there exist an orthonormal basis $\{e_i\}_{i=1}^m$ and a probability distribution $\{p_i\}_{i=1}^m$ with $p_i > 0$, such that $\omega(A) = \sum_{i=1}^m p_i \langle e_i | A e_i \rangle$ for all $A \in \mathcal{B}(\mathcal{H})$. Furthermore, $\omega(A) = \langle \Omega | (A \otimes \mathbb{1}) \Omega \rangle$, where $\Omega = \sum_{i=1}^m \sqrt{p_i} e_i \otimes e_i$. (This is, of course, the canonical Stinespring dilation of the CP map ω by means of the GNS construction.) Let $D_\omega \in \mathcal{B}(\mathcal{H})$ denote the density operator corresponding to ω , i.e., $\omega(A) = \text{Tr}(D_\omega A)$. Owing to the faithfulness of ω , D_ω is invertible.

Fix an orthonormal basis $\{f_\mu\}_{\mu=1}^n$ of \mathcal{K} , and define the channel $\Phi_\omega: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$ through $\Phi_\omega(A) = \omega(A) \mathbb{1}_{\mathcal{K}}$. The Kraus form of Φ_ω is given by $\Phi_\omega(A) = \sum_{i=1}^m \sum_{\mu=1}^n V_{i\mu}^* A V_{i\mu}$, where $V_{i\mu} = \sqrt{p_i} |e_i\rangle \langle f_\mu|$, and the canonical Stinespring dilation by $\Phi_\omega(A) = V_\omega^*(A \otimes \mathbb{1}_{\mathcal{E}})V_\omega$, where again $\mathcal{E} \simeq \mathcal{K} \otimes \mathcal{H}$ and $V_\omega \psi = \sum_{i=1}^m \sum_{\mu=1}^n V_{i\mu} \psi \otimes f_\mu \otimes e_i$.

Consider the positive operator $F_{T,\omega} = T \circ \text{id}((D_\omega^{-1} \otimes \mathbb{1})|\Omega\rangle\langle\Omega|(D_\omega^{-1} \otimes \mathbb{1}))$, whose matrix elements are given by $\langle f_\mu \otimes e_i | F_{T,\omega}(f_\nu \otimes e_j) \rangle = (1/\sqrt{p_i p_j}) \langle f_\mu | T(|e_i\rangle\langle e_j|) f_\nu \rangle$. For all $A \in \mathcal{B}(\mathcal{H})$ and $\psi \in \mathcal{K}$ we then have

$$\begin{aligned} V_\omega^*(A \otimes F_{T,\omega})V_\omega\psi &= \sum_{i,j=1}^m \sum_{\mu,\nu=1}^n \sqrt{p_i p_j} \langle e_i | A e_j \rangle \langle f_\mu \otimes e_i | F_{T,\omega}(f_\nu \otimes e_j) \rangle |f_\mu\rangle \langle f_\nu | \psi \rangle \\ &= \sum_{i,j=1}^m \sum_{\mu,\nu=1}^n \langle e_i | A e_j \rangle \langle f_\mu | T(|e_i\rangle \langle e_j|) f_\nu \rangle |f_\mu\rangle \langle f_\nu | \psi \rangle \equiv T(A)\psi, \end{aligned}$$

so that $T \leq \|F_{T,\omega}\| \Phi_\omega$, with $\|F_{T,\omega}\| \leq \|D_\omega^{-1}\|^2 \|T\|_{cb}$.

Consequently, for any faithful state ω on $\mathcal{B}(\mathcal{H})$ and any CP map $T: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$ there exists a positive constant c such that T is completely c -dominated by Φ_ω ; thus T is uniquely determined by the Radon–Nikodym derivative $D_{\Phi_\omega} T$. Note that in the special case of ω being the tracial state on \mathcal{M}_m we simply recover the results of the preceding section.

C. Generalization to infinite dimensions

In the form stated above, both the Jamiolkowski isomorphism and its generalization to arbitrary faithful states are valid only for CP maps between finite-dimensional algebras. However, in many problems of quantum information theory it is necessary to consider CP maps between algebras of operators on infinite-dimensional Hilbert spaces.

Consider a normal CP map $T: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$, where \mathcal{H} and \mathcal{K} are separable Hilbert spaces. Fix a normal faithful state ω on $\mathcal{B}(\mathcal{H})$; then there exist a complete orthonormal basis $\{e_i\}$ of \mathcal{H} and a probability distribution $\{p_i\}$, $p_i > 0$, such that, for any $A \in \mathcal{B}(\mathcal{H})$, $\omega(A) = \langle \Omega | (A \otimes \mathbb{1}) \Omega \rangle$ with $\Omega = \sum_i \sqrt{p_i} e_i \otimes e_i$. Let D_ω denote the density operator corresponding to ω . Because \mathcal{H} is infinite-dimensional, the inverse of D_ω is an unbounded operator defined on a dense domain, namely the linear span of $\{e_i\}$. Therefore the approach taken in the preceding section will not work; instead, we will characterize T through the Radon–Nikodym derivative of another CP map T_ω (dependent on both T and ω) with respect to the channel $\Phi_\omega = \omega(A) \mathbb{1}_{\mathcal{K}}$.

Choosing a complete orthonormal basis $\{f_\mu\}$ of \mathcal{K} , we can write Φ_ω in the Kraus form $\Phi_\omega(A) = \sum_{i,\mu} V_{i\mu}^* A V_{i\mu}$, $V_{i\mu} = \sqrt{p_i} |e_i\rangle \langle f_\mu|$, where the series converges in the strong operator topology. We also have the Stinespring dilation via $\Phi_\omega(A) = V_\omega^* (A \otimes \mathbb{1}_{\mathcal{E}}) V_\omega$, where $\mathcal{E} \cong \mathcal{K} \otimes \mathcal{H}$ and $V_\omega \psi = \sum_{i,\mu} V_{i\mu} \psi \otimes f_\mu \otimes e_i$. To see that this Stinespring dilation is canonical, let $A = (1/\sqrt{p_k}) |e_j\rangle \langle e_k|$ and $\psi = f_\nu$. Thus

$$(A \otimes \mathbb{1}_{\mathcal{E}}) V_\omega \psi = e_j \otimes f_\nu \otimes e_k,$$

which shows that the set $\{(A \otimes \mathbb{1}_{\mathcal{E}}) V_\omega \psi | A \in \mathcal{B}(\mathcal{H}), \psi \in \mathcal{K}\}$ is total in $\mathcal{H} \otimes \mathcal{E}$.

Let $F_{T,\omega} = T \otimes \text{id}(|\Omega\rangle \langle \Omega|)$; the matrix elements are

$$\langle f_\mu \otimes e_i | F_{T,\omega}(f_\nu \otimes e_j) \rangle = \sqrt{p_i p_j} \langle f_\mu | T(|e_i\rangle \langle e_j|) f_\nu \rangle.$$

Then for all $A \in \mathcal{B}(\mathcal{H})$ and $\psi \in \mathcal{K}$ we can write

$$\begin{aligned} V_\omega^*(A \otimes F_{T,\omega})V_\omega\psi &= \sum_{i,\mu} \sum_{j,\nu} \sqrt{p_i p_j} \langle f_\mu \otimes e_i | F_{T,\omega}(f_\nu \otimes e_j) \rangle \langle e_i | A e_j \rangle |f_\mu\rangle \langle f_\nu | \psi \rangle \\ &= \sum_{i,\mu} \sum_{j,\nu} p_i p_j \langle e_i | A e_j \rangle \langle f_\mu | T(|e_i\rangle \langle e_j|) f_\nu \rangle |f_\mu\rangle \langle f_\nu | \psi \rangle \equiv T(D_\omega A D_\omega) \psi. \end{aligned}$$

We will write $T_\omega(A)$ for $T(D_\omega A D_\omega)$. From the Radon–Nikodym theorem it follows that T_ω is completely dominated by Φ_ω , and that $D_{\Phi_\omega} T_\omega = \mathbb{1}_{\mathcal{H}} \otimes F_{T,\omega}$. We can determine the action of T on the “matrix units” $|e_i\rangle \langle e_j|$ via $T(|e_i\rangle \langle e_j|) = (p_i p_j)^{-1} T_\omega(|e_i\rangle \langle e_j|)$.

VI. NORM ESTIMATES FOR DIFFERENCES OF QUANTUM OPERATIONS

In this section we will demonstrate the use of the Radon–Nikodym theorem for CP maps in deriving several useful estimates for CB norms of differences of quantum channels.

Consider two CP maps $T_1, T_2: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$. Suppose that there exists a CP map $T: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$, such that $T_i \leq T$, $i = 1, 2$, and let $T(A) = V^*(A \otimes \mathbb{1}_{\mathcal{E}})V$ be the canonical Stinespring dilation of T . By the Radon–Nikodym theorem, there exist positive contractions $F_1, F_2 \in \mathcal{B}(\mathcal{E})$ such that $T_i(A) = V^*(A \otimes F_i)V$, $i = 1, 2$. Then

$$(T_1 - T_2)(A) = T_1(A) - T_2(A) = V^*(A \otimes (F_1 - F_2))V,$$

and the Haagerup–Paulsen–Wittstock theorem immediately implies that

$$\|T_1 - T_2\|_{cb} \leq \|V\| \|(F_1 - F_2)V\| \leq \|V\|^2 \|F_1 - F_2\|.$$

If T is a quantum channel, V is an isometry, so that $\|V\| = 1$. Therefore we get

$$\|T_1 - T_2\|_{cb} \leq \|F_1 - F_2\|. \tag{13}$$

In particular, if $S \leq T$, then $\|S - T\|_{cb} \leq \|\mathbb{1} - F\|$, where $\mathbb{1} \otimes F$ is the Radon–Nikodym derivative $D_T S$.

Given two CP maps $T_1, T_2: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$ with (not necessarily minimal) Stinespring dilations $T_i(A) = V_i^*(A \otimes \mathbb{1}_{\mathcal{E}})V_i$, $i = 1, 2$, on the common dilation space \mathcal{E} , the norm $\|T_1 - T_2\|_{cb}$ can be bounded from above in terms of V_1 and V_2 . Indeed, denoting by π the $*$ -homomorphism $\mathcal{B}(\mathcal{H}) \ni A \mapsto A \otimes \mathbb{1}_{\mathcal{E}}$, we can use the Haagerup–Paulsen–Wittstock theorem to obtain

$$\begin{aligned} \|T_1 - T_2\|_{cb} &= \|V_1^* \circ \pi \circ V_1 - V_2^* \circ \pi \circ V_2\|_{cb} \\ &\leq \|V_1^* \circ \pi \circ V_1 - V_1^* \circ \pi \circ V_2\|_{cb} + \|V_1^* \circ \pi \circ V_2 - V_2^* \circ \pi \circ V_2\|_{cb} \\ &\leq (\|V_1\| + \|V_2\|) \|V_1 - V_2\|. \end{aligned} \tag{14}$$

If T_1 and T_2 are channels, then V_1 and V_2 are isometries. Consequently, $\|V_1\| = \|V_2\| = 1$, and the bound (14) becomes $\|T_1 - T_2\|_{cb} \leq 2\|V_1 - V_2\|$. As the lemma below shows, when the Hilbert spaces \mathcal{H} and \mathcal{K} are finite-dimensional, one can find a common dilation space \mathcal{E} and maps $V_1, V_2: \mathcal{K} \rightarrow \mathcal{H} \otimes \mathcal{E}$, such that $\|T_1 - T_2\|_{cb}$ can be bounded from below.

Lemma VI.1: For any two CP maps $T_1, T_2: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$ there exist a Hilbert space \mathcal{E} and operators $V_1, V_2: \mathcal{K} \rightarrow \mathcal{H} \otimes \mathcal{E}$ such that $T_i(A) = V_i^*(A \otimes \mathbb{1}_{\mathcal{E}})V_i$, $i = 1, 2$, and

$$\|V_1 - V_2\| \leq \dim \mathcal{H} \sqrt{\|T_1 - T_2\|_{cb}}. \tag{15}$$

Proof: Using Theorem V.1, we can write $\mathcal{E} = \mathcal{K} \otimes \mathcal{H}$ and $V_i = \sqrt{D_{\Phi} T_i} V_{\Phi} = (\mathbb{1}_{\mathcal{H}} \otimes \sqrt{F_{T_i}}) V_{\Phi}$. Then $T_i(A) = V_i^*(A \otimes \mathbb{1}_{\mathcal{E}})V_i$. Next we prove the estimate (15). We have

$$\|V_1 - V_2\| \leq \|\mathbb{1}_{\mathcal{H}} \otimes \sqrt{F_{T_1}} - \mathbb{1}_{\mathcal{H}} \otimes \sqrt{F_{T_2}}\| \|V_{\Phi}\| = \|\sqrt{F_{T_1}} - \sqrt{F_{T_2}}\| \leq \sqrt{\|F_{T_1} - F_{T_2}\|}. \tag{16}$$

The last inequality in (16) holds because: (1) $x \mapsto \sqrt{x}$ is an operator monotone function on $[0, \infty)$, i.e., $\sqrt{A} \leq \sqrt{B}$ for all operators A, B satisfying $0 \leq A \leq B$ (Prop. V.1.8 in Ref. 38), (2) for any operator monotone function f with $f(0) = 0$ and any pair of positive operators A, B we have $\|f(A) - f(B)\| \leq f(\|A - B\|)$ (Theorem X.1.1 in Ref. 38), and (3) $\|X\| = \|\sqrt{X}\|^2$ for any $X \geq 0$ by the spectral mapping theorem. Now $F_{T_i} = (\dim \mathcal{H})^2 T_i \otimes \text{id}(|\Psi\rangle\langle\Psi|)$, where $\Psi = (1/\sqrt{\dim \mathcal{H}}) \sum_i e_i \otimes e_i$ for some orthonormal basis $\{e_i\}$ in \mathcal{H} . Thus, using the properties of the CB norm, we get

$$\|F_{T_1} - F_{T_2}\| = (\dim \mathcal{H})^2 \|T_1 \otimes \text{id}(|\Psi\rangle\langle\Psi|) - T_2 \otimes \text{id}(|\Psi\rangle\langle\Psi|)\| \leq (\dim \mathcal{H})^2 \|T_1 - T_2\|_{cb}. \tag{17}$$

Combining Eqs. (16) and (17) yields (15). ■

Inequality (15) was also proved by Kitaev,²¹ but by quite different means. Here several warnings are in order. In the article of Kitaev²¹ the “canonical representation” of a CP map $T: \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K})$ is defined as $T(A) = \text{Tr}_{\mathcal{F}} W A W^*$ with $\mathcal{F} = \mathcal{K} \otimes \mathcal{H}$. This is not to be confused with the *canonical Stinespring dilation* of T , $T(A) = V^*(A \otimes \mathbb{1}_{\mathcal{E}})V$ [or its dual, $T_*(A) = \text{Tr}_{\mathcal{E}} V A V^*$] which must satisfy the requirement that $\mathcal{H} \otimes \mathcal{E}$ is (the closure of) the linear span of $\{(A \otimes \mathbb{1}_{\mathcal{E}})V\psi \mid A \in \mathcal{B}(\mathcal{H}), \psi \in \mathcal{K}\}$. Thus \mathcal{E} is, in general, a subspace of $\mathcal{F} = \mathcal{K} \otimes \mathcal{H}$. Furthermore, Kitaev’s version of the estimate (15) has $\dim \mathcal{K}$, and not $\dim \mathcal{H}$, multiplying the CB norm on its right-hand side. This is due to the fact that, whereas we cast all CP maps in the Stinespring form $T(A) = W^*(A \otimes \mathbb{1}_{\mathcal{F}})W$, Kitaev prefers to work with the dual representation $T_*(A) = \text{Tr}_{\mathcal{F}} W A W^*$. Since all (bounded) operators on a finite-dimensional Hilbert space are trace-class, T_* trivially extends to a CP map from $\mathcal{B}(\mathcal{K})$ into $\mathcal{B}(\mathcal{H})$.

VII. CONCLUDING REMARKS

In this paper we have shown that the Radon–Nikodym theorem for completely positive maps^{6,7,10} is an extremely powerful and versatile tool for problems involving characterization and comparison of quantum operations. The upshot is that if $T(A) = V^*(A \otimes \mathbb{1}_{\mathcal{E}})V$ is the canonical Stinespring dilation of a CP map T , then the set of all CP maps S for which $T - S$ is also CP (we say that S is *completely dominated* by T) is in a one-to-one correspondence with the positive contractions F on \mathcal{E} , given explicitly by $S(A) = V^*(A \otimes F)V$. As we have demonstrated, this correspondence brings many seemingly unrelated problems into a common framework.

However, many important questions still remain unanswered. For instance, it is not difficult to convert the above “Stinespring form” of the Radon–Nikodym theorem into an equivalent “Kraus form” (cf. Sec. III B). The Kraus decomposition of a CP map T involves at most countably many terms, and all maps S completely dominated by T can be characterized in terms of positive-definite kernels on the corresponding indexing set. However, it is not clear how to apply this theorem directly to CP maps given in terms of a “continual” Kraus decomposition (as in, e.g., the quantum operational model of Gaussian displacement noise³⁹). For example, if U_g is a strongly continuous unitary representation of a compact topological group G on a Hilbert space \mathcal{H} , how do we describe all CP maps completely dominated by the channel

$$T(A) = \int_G U_g^* A U_g \, d\mu(g),$$

where μ is the (normalized) Haar measure on G , in terms of $\{U_g\}$? A partial step in this direction has been taken by Parthasarathy,¹⁰ who constructed a Stinespring dilation of T in terms of $\{U_g\}$ under the assumption that these operators are linearly independent μ -almost everywhere, i.e.,

$$\int_G \varphi(g) U_g \, d\mu(g) = 0 \Leftrightarrow \varphi(g) = 0 \quad \mu - \text{a.e.}$$

for any $\varphi \in \mathcal{L}^1(G, \mu)$. However, a general solution is still lacking. We hope to address this issue in a future publication.

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