Abstract—The Blackwell measure of a binary-input channel (BIC) is the distribution of the posterior probability of 0 under the uniform input distribution. This paper gives an explicit characterization of the evolution of the Blackwell measure of an arbitrary symmetric BIC under Arıkan’s polarization transform, and uses this characterization to provide a unifying set of techniques for studying the polarization phenomenon.

Index Terms—Channel polarization, Blackwell measure, convex functions of channels.

I. INTRODUCTION

Polar codes, introduced by Arıkan [1], are a structured family of codes with low encoding and decoding complexity that provably achieve capacity for any symmetric binary-input channel (BIC). The salient feature of polar codes, from which they derive their name, is the polarization phenomenon: $n$ independent copies of a given symmetric BIC are transformed in such a way that most of the resulting $n$ channels are either nearly completely noisy (with capacity near zero) or nearly completely noiseless (with capacity near 1 bit).

In the original paper of Arıkan [1], the polarization phenomenon was demonstrated for two channel parameters: the capacity and the Bhattacharyya parameter. Subsequent results by Alsan and Telatar [2]–[4] have revealed other channel parameters that polarize – for example, Gallager’s $E_0$ function, which is related to various error exponents and cutoff rates. However, the proofs of polarization are usually tailored to the particular channel parameter of interest, and there does not appear to be a common technique underlying them.

The objective of this paper is to offer such a technique that relies on viewing each BIC through the lens of its Blackwell measure [5], [6], i.e., the distribution of the posterior probability of input symbol being 0 under the uniform input distribution. We exploit two key facts pertaining to this representation: (a) the Blackwell measure of BSC($p$) assigns equal masses to the points $p$ and $1-p,$ and (b) each symmetric BIC can be decomposed into a mixture of BSCs with input-independent weights. Using these facts, we can easily track the evolution of the Blackwell measures under the polarization transform, which in turn gives us a handle on the polarization phenomenon for a very broad class of channel characteristics.

Notation. For $p, q \in [0, 1],$ we let $\bar{p} = 1-p$ (for $p \in \{0, 1\},$ this is the Boolean NOT) and $p \times q = \bar{p}q + pq.$ For $a, b \in \mathbb{R},$ we let $a \wedge b \triangleq \min(a, b)$ and $a \vee b \triangleq \max(a, b).$ Given a random object $U,$ we will denote by $\mathcal{L}(U)$ its probability law.

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This work was supported in part by Center for Science of Information (CSoI), an NSF Science and Technology Center, under grant agreement CCF-0939370.

II. BICS AND THEIR BLACKWELL MEASURES

A discrete binary-input channel (BIC) is a pair $(Y, W),$ where $Y$ is the finite output alphabet and $W = (W(\cdot|0), W(\cdot|1))$ is a pair of probability distributions on $Y.$ For $x \in \{0, 1\}, W(\cdot|x)$ is the probability distribution of the channel output when the channel input is equal to $x.$ Given two BICS $(Y_1, W_1)$ and $(Y_2, W_2),$ we define the product BIC $(Y_1 \times Y_2, W_1 \times W_2)$ by

$$(W_1 \times W_2)(y_1, y_2|x) = W_1(y_1|x)W_2(y_2|x)$$

for all $x \in \{0, 1\}$ and all $(y_1, y_2) \in Y_1 \times Y_2.$ In other words, $W_1 \times W_2$ is the parallel broadcast channel formed by $W_1$ and $W_2$ [7]. A BIC $(Y, W)$ is symmetric [1] if there exists a bijection $\pi : Y \rightarrow Y,$ such that $\pi^{-1} = \pi$ and $W(\pi y|0) = W(y|1)$ for all $y \in Y.$

Our analysis hinges on a certain representation of channels that originates in the work of Blackwell on the comparison of statistical experiments [5], [6] (see [8] for a modern synthesis). We particularize it to BICs as follows. Given a BIC $(Y, W),$ let $(X, Y)$ be a random couple with $P_X = \text{Bern}(1/2)$ and $P_{Y|X} = W.$ The random variable

$$S = S_W(Y) \triangleq \frac{W(Y|0)}{W(Y|0) + W(Y|1)},$$

which is equal to the posterior probability of $X = 0$ given $Y,$ takes values in the unit interval $[0, 1]$ and has mean 1/2. The Blackwell measure of $W,$ which we will denote by $m_W,$ is the probability law of $S.$ For example:

$$m_{\text{BSC}(p)} = \frac{1}{2} \delta_p + \frac{1}{2} \delta_{\bar{p}}, \quad m_{\text{BEC}(\varepsilon)} = \frac{\varepsilon}{2} \delta_0 + \frac{\varepsilon}{2} \delta_1 + \varepsilon \delta_{1/2},$$

where $\delta_a$ denotes the Dirac measure at $a.$ Given two BICS $(Y, W)$ and $(Y, W'),$ we say that $W$ dominates $W'$ (or is more informative than $W'$) if there exists a random transformation $K$ from $Y$ to $Y',$ such that $W' = K \circ W$ (i.e., $W'$ is stochastically deprecated with respect to $W$). In that case, we write $W \geq W'.$ We say that $W$ and $W'$ are equivalent if $W \geq W'$ and $W' \geq W.$ In that case, we write $W \equiv W'.$

Theorem 1 (Blackwell–Sherman–Stein). Consider two BICS $W$ and $W'.$ Then:

1) $W \equiv W'$ if and only if $m_W = m_{W'}$ (that is, the Blackwell measure specifies the channel uniquely up to equivalence). Moreover, let $\mathcal{M}$ denote the collection of all Borel probability measures on $[0, 1]$ with mean 1/2. Then for any $m \in \mathcal{M}$ there exists a BIC $W,$ unique up to equivalence, such that $m = m_W.$

$^1$The BIC $W$ has a finite output alphabet if and only if $m_W$ has finite support. This is precisely the setting of this paper.
where $S$ is expressed in this way; for example:

\[ B_{\text{symmetric}} \]

also evident from the decomposition $B_{\text{symmetric}}(\text{BIC})$ is a mixture of Blackwell measures of BSCs. Thus, Theorem 2 shows that the Blackwell measure of any measures of the constituent subchannels: compound channel is given by the mixture of the Blackwell measures with parameters $P_{\text{ML}}$, for any symmetric BIC $B_{\text{symmetric}}(\text{BIC})$.

Theorem 2:

For any symmetric BIC $B_{\text{symmetric}}(\text{BIC})$, there exist a positive integer $n$, a probability vector $\lambda = (\lambda_1, \ldots, \lambda_n)$, and error parameters $p_1, \ldots, p_n \in [0, 1]$, such that

\[ W \equiv \sum_{i=1}^{n} \lambda_i \text{BSC}(p_i). \]

(1)

It is not difficult to show that the Blackwell measure of a compound channel is given by the mixture of the Blackwell measures of the constituent subchannels:

\[ m_W = \sum_{i=1}^{m} \lambda_i m_{W_i}. \]

Thus, Theorem 2 shows that the Blackwell measure of any symmetric BIC is a mixture of Blackwell measures of BSCs. For example, $BEC(\varepsilon) \equiv \varepsilon \text{BSC}(0) \oplus \varepsilon \text{BSC}(1/2)$, which is also evident from the decomposition $m_{\text{BEC}(\varepsilon)} = \varepsilon m_{\text{BSC}(0)} + \varepsilon m_{\text{BSC}(1/2)}$.

### A. Functional properties

Any measurable function $f : [0, 1] \rightarrow \mathbb{R}$ induces a functional $I_f$ on the collection of all BICs via

\[ I_f(W) \equiv \int_{[0, 1]} f \, dm_W = \mathbb{E}[f(S)], \]

where $S \sim m_W$. A variety of channel characteristics can be expressed in this way; for example:

- With $f(s) = \log 2 - h_2(s)$, where $h_2(\cdot)$ is the binary entropy function, $I_f(W)$ is equal to the symmetric capacity $I(W)$ of $W$ [1], i.e., the mutual information $I(p, W)$ with $P = \text{Bern}(1/2)$.
- With $f(s) = 2\sqrt{s(1-s)}$, we get

\[ I_f(W) = Z(W) \equiv \sum_{y \in Y} \sqrt{W(y|0)W(y|1)}, \]

the Bhattacharyya parameter of $W$ [1]. More generally, with $f(s) = 2s^\alpha(1-s)^{1-\alpha}$ for $\alpha \in [0, 1]$, $I_f(W)$ is equal to the Hellinger affinity of order $\alpha$:

\[ H_\alpha(W) \triangleq \sum_{y \in Y} W(y|0)^\alpha W(y|1)^{1-\alpha}. \]

- Taking

\[ f(s) = 2^{-\rho} \left( s^{1/\alpha} + (1-s)^{1/\alpha} \right)^{1+\rho} \]

for some $\rho \geq 0$, we can relate $I_f(W)$ to Gallager’s function $E_0(p, W) = E_0(p, \text{Bern}(1/2), W)$ [12] evaluated at the uniform input distribution:

\[ I_f(W) = \exp \left( -E_0(p, W) \right). \]

With $f(s) = 2s - 1$, we get $I_f(W) = 1 - 2P_{\text{ML}}(W)$, where $P_{\text{ML}}(W)$ denotes the probability of error of maximum-likelihood decoding of a single equiprobable bit sent through the channel $W$ [4, Ch. 5]. With $f_\lambda(s) = \lambda \wedge (\lambda s) \wedge (\lambda s)$ for $0 < \lambda < 1$, $I_f(W)$ is equal to

\[ B_\lambda(W) \equiv \frac{1}{2}|1 - 2\lambda| - \frac{1}{2}\|W(\bullet|1) - W(\bullet|0)\|_{TV}, \]

the Bayesian information gain about $X \sim \text{Bern}(\lambda)$ based on a single observation through $W$. Since any convex function $f : [0, 1] \rightarrow \mathbb{R}$ can be approximated by a positive linear combination of such $f_\lambda$’s [8], it follows that $W \succeq W'$ if and only if $B_\lambda(W) \geq B_\lambda(W')$ for all $\lambda \in (0, 1)$.

It follows directly from definitions that

\[ I_f(\text{BSC}(p)) = \frac{1}{2} f(p) + \frac{1}{2} f(p^\ast). \]

Moreover, if $W$ is a symmetric BIC that admits the decomposition (1), then

\[ I_f(W) = \sum_{i=1}^{n} \lambda_i I_f(\text{BSC}(p_i)) = \sum_{i=1}^{n} \frac{\lambda_i [f(p_i) + f(p_i^\ast)]}{2}. \]

### B. Operations on Blackwell measures

Given two Blackwell measures $m_1, m_2 \in \mathcal{M}$, we define two probability measures $m_1 \oplus m_2$ and $m_1 \odot m_2$ on $[0, 1]$ as follows. Let $S_1 \sim m_1$ and $S_2 \sim m_2$ be two independent random variables. For any continuous $f : [0, 1] \rightarrow \mathbb{R}$, let

\[ \int_{[0, 1]} f \, dm_1 \oplus m_2 = \mathbb{E}[f(1 - S_1 \ast S_2)] \]

and

\[ \int_{[0, 1]} f \, dm_1 \odot m_2 = \mathbb{E} \left[ (1 - S_1 \ast S_2) f \left( \frac{S_1 S_2}{1 - S_1 \ast S_2} \right) + (S_1 \ast S_2) f \left( \frac{S_1 S_2}{S_1 \ast S_2} \right) \right]. \]

Since the interval $[0, 1]$ is compact, every continuous $f : [0, 1] \rightarrow \mathbb{R}$ is bounded by the Weierstrass theorem.
Lemma 1. The probability measures \( m_1 \oplus m_2 \) and \( m_1 \odot m_2 \) are also Blackwell measures.

Proof. Using \( f(s) = s \) in Eqs. (2) and (3), and recalling that \( S_1 \) and \( S_2 \) are independent and both have mean \( \frac{1}{2} \), we get

\[
\int_{[0,1]} s(m_1 \oplus m_2)(ds) = \mathbb{E}[1 - S_1 \star S_2] = \frac{1}{2}
\]

and

\[
\int_{[0,1]} s(m_1 \odot m_2)(ds) = \mathbb{E}[S_2] = \frac{1}{2}.
\]

Thus, both \( m_1 \oplus m_2 \) and \( m_1 \odot m_2 \) are in \( M \).

III. EVOLUTION OF BLACKWELL MEASURES UNDER THE POLARIZATION TRANSFORM

The polarization transform maps a pair of BICs \( (Y_1, W_1) \) and \( (Y_2, W_2) \) into another pair of BICs \( (Y_1 \times Y_2, W_1 \oplus W_2) \) and \( (Y_1 \times Y_2 \times \{0, 1\}, W_1 \circ W_2) \) as follows:

\[
(W_1 \oplus W_2)(y_1, y_2|x) \triangleq \frac{1}{2} \sum_{u \in \{0, 1\}} W_1(y_1|u \oplus x)W_2(y_2|u) \tag{4a}
\]

\[
(W_1 \circ W_2)(y_1, y_2, u|x) \triangleq \frac{1}{2} W_1(y_1|u \oplus x)W_2(y_2|x) \tag{4b}
\]

for all \( x, u \in \{0, 1\} \) and all \( (y_1, y_2) \in Y_1 \times Y_2 \), where \( \oplus \) is the Boolean XOR. The Blackwell measures of \( W_1 \oplus W_2 \) and \( W_1 \circ W_2 \) can be computed from those of \( W_1 \) and \( W_2 \):

Theorem 3. The Blackwell measures of the BICs \( W_1 \oplus W_2 \) and \( W_1 \circ W_2 \) are given by

\[
m_{W_1 \oplus W_2} = m_{W_1} \oplus m_{W_2} \quad \text{and} \quad m_{W_1 \circ W_2} = m_{W_1} \odot m_{W_2}.
\]

Remark 2. This result can be derived as a special case of Prop. 3 in [10], but we give a self-contained proof below.

Proof. The formula for \( W_1 \oplus W_2 \) follows from a straightforward computation. Now, from the definition (4) it follows that

\[
W_1 \circ W_2 \equiv \frac{1}{2} W^{(0)} \oplus \frac{1}{2} W^{(1)} \tag{5}
\]

with \( W^{(0)} \triangleq W_1 \times W_2 \) and \( W^{(1)} \triangleq W_1 \circ W_2 \), where \( W_1 \) is the BIC related to \( W_1 \) via \( W_1(\cdot|x) = W_1(\cdot|x) \). Then the random variables \( S^{(s)} \sim m_{W^{(s)}}, s \in \{0, 1\} \), evidently satisfy

\[
\mathbb{E}[f(S^{(0)})] = 2 \mathbb{E} \left[ (1 - S_1 \star S_2)f \left( \frac{S_1S_2}{1 - S_1 \star S_2} \right) \right]
\]

and

\[
\mathbb{E}[f(S^{(1)})] = 2 \mathbb{E} \left[ (S_1 \star S_2)f \left( \frac{S_1S_2}{S_1 \star S_2} \right) \right]
\]

for every continuous \( f : [0, 1] \to \mathbb{R} \). From this and (5), we get

\[
\int_{[0,1]} f dm_{W_1 \oplus W_2} = \frac{1}{2} \mathbb{E}[f(S^{(0)})] + \frac{1}{2} \mathbb{E}[f(S^{(1)})] = \mathbb{E} \left[ (1 - S_1 \star S_2)f \left( \frac{S_1S_2}{1 - S_1 \star S_2} \right) + (S_1 \star S_2)f \left( \frac{S_1S_2}{S_1 \star S_2} \right) \right] = \int_{[0,1]} f dm_{W_1 \circ W_2}.
\]

Since \( f \) is arbitrary, we obtain the formula for \( m_{W_1 \oplus W_2} \).

Using this theorem, we can explicitly write down the image of a pair of BSCs under the polarization transformation:

**Corollary 1** (Polarization of BSCs). For any pair of crossover probabilities \( (p, q) \in [0, \frac{1}{2}] \times [0, \frac{1}{2}] \),

\[
\text{BSC}(p) \oplus \text{BSC}(q) \equiv \text{BSC}(p \star q)
\]

and

\[
\text{BSC}(p) \odot \text{BSC}(q) \equiv \text{BSC}(p) \times \text{BSC}(q).
\]

**Proof.** A random variable \( S \sim m_{\text{BSC}(p)} \) takes two equiprobable values, \( p \) and \( \bar{p} \). Therefore, if \( S_1 \sim \text{BSC}(p) \) and \( S_2 \sim \text{BSC}(q) \) are independent, the random variable \( 1 - S_1 \star S_2 \) takes two equiprobable values, \( p \times q \) and \( 1 - p \times q = \bar{p} \times q \) --- that is,

\[
m_{\text{BSC}(p) \oplus \text{BSC}(q)} = m_{\text{BSC}(p)} \oplus m_{\text{BSC}(q)} = \frac{1}{2} \delta_{p \times q} + \frac{1}{2} \delta_{1 - p \times q}
\]

which proves (6). Next, from the definition (3), for any continuous \( f : [0, 1] \to \mathbb{R} \) we have

\[
\int_{[0,1]} f dm_{\text{BSC}(p) \odot \text{BSC}(q)} = \frac{1}{2} \left( 1 - p \times q \right) f \left( \frac{pq}{1 - p \times q} \right) + \frac{1}{2} \left( 1 - p \times q \right) f \left( \frac{\bar{p}q}{1 - p \times q} \right) + \frac{1}{2} \left( p \times q \right) f \left( \frac{pq}{p \times q} \right) + \frac{1}{2} \left( p \times q \right) f \left( \frac{\bar{p}q}{p \times q} \right) - \frac{1}{2} \left( p \times q \right) f \left( \frac{pq}{p \times q} \right) + \frac{1}{2} \left( p \times q \right) f \left( \frac{\bar{p}q}{p \times q} \right) .
\]

On the other hand, a straightforward computation shows that

\[
m_{\text{BSC}(p) \times \text{BSC}(q)} = (p \times q) \left( \frac{1}{2} \delta_{p \times q} + \frac{1}{2} \delta_{\bar{p} \times q} \right) + (1 - p \times q) \left( \frac{1}{2} \delta_{p \times q} + \frac{1}{2} \delta_{\bar{p} \times q} \right),
\]

which proves (7).

We can also characterize the image of a pair of arbitrary symmetric BICs under the polarization transformation:

**Corollary 2.** For any pair \( W_1 \) and \( W_2 \) of symmetric BICs, there exist positive integers \( m, n \), probability vectors
\[ \lambda = (\lambda_1, \ldots, \lambda_n), \mu = (\mu_1, \ldots, \mu_n), \text{ and tuples } p = (p_1, \ldots, p_m) \in [0,1]^m, q = (q_1, \ldots, q_n) \in [0,1]^n, \text{ such that} \]
\[
m_{W_1 \otimes W_2} = \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_i \mu_j m_{\text{BSC}}(p_i \otimes q_j)
\]
and
\[
m_{W_1 \oplus W_2} = \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_i \mu_j m_{\text{BSC}(p_i) \times \text{BSC}(q_j)}.
\]

\textbf{Proof.} From Theorem 2, we can write
\[
m_{W_1} = \sum_{i=1}^{m} \lambda_i m_{\text{BSC}(p_i)} = \sum_{i=1}^{m} \lambda_i (\delta_{p_i} + \delta_{\bar{p}_i})/2
\]
and
\[
m_{W_2} = \sum_{j=1}^{n} \mu_j m_{\text{BSC}(q_j)} = \sum_{j=1}^{n} \mu_j (\delta_{q_j} + \delta_{\bar{q}_j})/2
\]
for some choices of \((m, \lambda, p)\) and \((n, \mu, q)\). Therefore, for two independent r.v.’s \(S_1 \sim m_{W_1}\) and \(S_2 \sim m_{W_2}\) and for any continuous \(f : [0, 1] \rightarrow \mathbb{R}\) we have
\[
\int_{[0,1]} f \, dm_{W_1 \otimes W_2} = \mathbb{E}[f(1 - S_1 \ast S_2)]
\]
\[
= \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_i \mu_j \left( \frac{1}{2} f(p_i \ast q_j) + \frac{1}{2} f(1-p_i \ast q_j) \right)
\]
and
\[
\int_{[0,1]} f \, dm_{W_1 \oplus W_2} = \mathbb{E} \left[ (1 - S_1 \ast S_2) f \left( \frac{S_1 S_2}{1-S_1 \ast S_2} \right) \right. \\
+ (S_1 \ast S_2) f \left( \frac{S_1 S_2}{S_1 \ast S_2} \right) \right]
\]
\[
= \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_i \mu_j \left( \frac{1}{2} (1-p_i \ast q_j) f \left( \frac{p_i q_j}{1-p_i \ast q_j} \right) \\
+ \frac{1}{2} (p_i \ast q_j) f \left( \frac{p_i q_j}{p_i \ast q_j} \right) + \frac{1}{2} (p_i \ast q_j) f \left( \frac{p_i q_j}{p_i \ast q_j} \right) \right).
\]

Applying Corollary 1, we obtain Eqs. (8) and (9).

\textbf{IV. POLARIZATION IS GENERIC}

Informally speaking, the polar transform (4) replaces the original pair of BICs \(W_1\) and \(W_2\) with another pair, where one BIC \(W_1 \otimes W_2\) is “better” than \(W_1\) or \(W_2\) and another BIC \(W_1 \oplus W_2\) is “worse” than both \(W_1\) and \(W_2\). This is precisely the polarization effect that is responsible for the capacity-achieving performance of polar codes. To be more precise, let us fix a class \(\mathcal{W}\) of BICs and a functional \(\Phi\) that associates a real number \(\Phi(W)\) to each BIC \(W\). Then we will say that \(\Phi\) polarizes on \(W\) if, for any two BICs \(W_1, W_2 \in \mathcal{W}\),
\[
\Phi(W_1 \otimes W_2) \leq \Phi(W_1) \land \Phi(W_2)
\]
\[
\leq \Phi(W_1) \lor \Phi(W_2) \leq \Phi(W_1 \oplus W_2). \tag{10}
\]

Of course, this property has an operational meaning only if \(\Phi(W) \geq \Phi(W')\) corresponds to \(W\) being somehow “better” than \(W'\). For example, Arikan has shown that the functionals \(\Phi(W) = I(W)\) and \(\Phi(W) = -Z(W)\) polarize [1]. Since then, the polarization property has been demonstrated for other channel parameters, such as Gallager’s \(E_0\) [4]. We now show that the polarization phenomenon is generic, i.e., Eq. (10) holds for a very broad class of channel functionals:

\textbf{Theorem 4. All channel functionals} \(I_f\) \textbf{with a convex} \(f : [0, 1] \rightarrow \mathbb{R}\) \textbf{polarize on the class of all symmetric BICs. That is, if} \(W_1, W_2\) \textbf{are two symmetric BICs, then}
\[
I_f(W_1 \otimes W_2) \leq I_f(W_1) \land I_f(W_2)
\]
\[
\leq I_f(W_1) \lor I_f(W_2) \leq I_f(W_1 \oplus W_2).
\]

\textbf{Proof.} Let \(S_1 \sim m_{W_1}\) and \(S_2 \sim m_{W_2}\) be independent. Then, using Theorem 3, we can write
\[
I_f(W_1 \otimes W_2) = \int_{[0,1]} f \, dm_{W_1 \otimes W_2}
\]
\[
= \int_{[0,1]} f \, dm_{W_1} \otimes m_{W_2}
\]
\[
= \mathbb{E}[f(S_1 S_2 + (1 - S_1)(1 - S_2))]
\]
\[
= \mathbb{E}[f(S_1 S_2 + (1 - S_1)(1 - S_2))]/S_2]\]
\[
\leq \mathbb{E}[S_2 f(S_1) + (1 - S_2) f(1 - S_1)]
\]
\[
\geq \frac{1}{2} \mathbb{E}[f(S_1)] + \frac{1}{2} \mathbb{E}[f(1 - S_1)]
\]
\[
= I_f(W_1),
\]
where (a) is by Jensen’s inequality, (b) follows from the fact that \(S_1\) and \(S_2\) are independent with \(\mathbb{E}[S_1] = \mathbb{E}[S_2] = \frac{1}{2}\), and (c) follows from the symmetry of \(W_1\), which is equivalent to \(\mathcal{L}(S_1) = \mathcal{L}(1 - S_1)\). This shows that \(I_f(W_1 \otimes W_2) \leq I_f(W_1)\). Conditioning on \(S_1\) instead of \(S_2\), we prove that \(I_f(W_1 \oplus W_2) \leq I_f(W_2)\).

Using Theorem 3 and Jensen’s inequality, we obtain
\[
I_f(W_1 \otimes W_2) = \int_{[0,1]} f \, dm_{W_1 \otimes W_2}
\]
\[
= \int_{[0,1]} f \, dm_{W_1} \otimes m_{W_2}
\]
\[
= \mathbb{E}[f(S_1 S_2)]
\]
\[
\geq \mathbb{E}[S_2 f(S_1)]
\]
\[
= I_f(W_1).
\]
By symmetry, the channels \(W_1 \otimes W_2\) and \(W_2 \otimes W_1\) are equivalent, so we also have \(I_f(W_1 \otimes W_2) \geq I_f(W_1)\).

\textbf{Corollary 3.}
\[
W_1 \otimes W_2 \preceq B W_1 \preceq B W_1 \otimes W_2
\]
\[
W_1 \otimes W_2 \preceq B W_2 \preceq B W_1 \otimes W_2.
\]
Theorem 5. The polarization process is naturally linked to channel domination in the sense of Blackwell. In particular, it implies the polarization phenomenon for the Bayes error functionals $B_3(\cdot)$ (see Sec. II-A), which, to the best of our knowledge, has not been discussed previously.

V. THE SUBMARTINGALE PROPERTY

Another salient feature of the polarization process noted by Arıkan [1] is the following: Fix a BIC $W$ and let $\{B_n\}_{n=1}^{\infty}$ be a sequence of i.i.d. Bern($\frac{1}{2}$) r.v.'s. Let $W_0 = W$, and

$$W_n = \begin{cases} W_{n-1} \oplus W_{n-1}, & \text{if } B_n = 0 \\ W_{n-1} \ominus W_{n-1}, & \text{if } B_n = 1 \end{cases}$$

for $n \geq 1$. Define the random processes $\{I_n\}_{n=0}^{\infty}$ and $\{Z_n\}_{n=0}^{\infty}$ via $I_n = I(W_n)$ and $Z_n = Z(W_n)$. Then $\{I_n\}$ is a nonnegative martingale, while $\{Z_n\}$ is a nonnegative supermartingale, both with respect to the natural filtration generated by $\{B_n\}$. These properties are a consequence of the following relations: for any pair $W_1, W_2$ of symmetric BICs,

$$I(W_1 \oplus W_2) + I(W_1 \ominus W_2) = I(W_1) + I(W_2)$$

and

$$Z(W_1 \oplus W_2) + Z(W_1 \ominus W_2) \leq Z(W_1) + Z(W_2).$$

Motivated by this, we introduce the following definition: Let $f : [0,1] \rightarrow \mathbb{R}$ be a convex function. We say that the polarization process is $f$-improving on a given collection $W$ of BICs if, for any $W_1, W_2 \in W$,

$$I_f(W_1 \oplus W_2) + I_f(W_1 \ominus W_2) \geq I_f(W_1) + I_f(W_2). \quad (11)$$

If (11) holds, then the random process $\{I_n^{(f)}\}_{n=0}^{\infty}$ with $I_n^{(f)} = I_f(W_n)$ is a submartingale. The following result shows that, in order to establish (11) for all symmetric BICs, it suffices to verify it on the subclass consisting only of BSCs:

Theorem 5. The polarization process is $f$-improving on the class of all symmetric BICs if and only if (11) holds for all pairs $(W_1, W_2) = (\text{BSC}(p), \text{BSC}(q))$, $(p, q) \in [0, \frac{1}{2}] \times [0, \frac{1}{2}]$.

Proof. If (11) holds for all symmetric BICs, then it holds for all BSCs. To prove the converse, fix two symmetric BICs $W_1, W_2$. By Theorem 2 we can write

$$W_1 = \bigoplus_{i=1}^{m} \lambda_i \text{BSC}(p_i) \quad \text{and} \quad W_2 = \bigoplus_{j=1}^{n} \mu_j \text{BSC}(q_j).$$

By Corollary 2,

$$m_{W_1 \oplus W_2} = \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_i \mu_j m_{\text{BSC}(p_i \oplus q_j)}$$

and

$$m_{W_1 \ominus W_2} = \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_i \mu_j m_{\text{BSC}(p_i \ominus q_j)}.$$